Repeated Contests with Private Information

(Job Market Paper)

Greg Kubitz

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Abstract

In repeated contests with private information, weak contestants prefer to appear strong while strong contestants prefer to appear weak. In contrast to a single contest, this leads to an equilibrium where effort is not strictly monotonic in ability and allows for a less able contestant to win against a contestant of higher ability. While the aggregate payoffs of contestants are higher per contest than in the single contest benchmark, aggregate output per contest is lower. Depending on the economic setting, the presence of private information can lead to productive or allocational inefficiencies.

1 Introduction

Contests are frequently used to stimulate effort from economic agents. These contests are often dynamic and offer multiple prizes, as in the case, for example, of innovation races and employee competitions. In both of these settings, there is extensive literature discussing how to best design contests to maximize the output of the contestants.[1] However, the behavior of economic agents in repeated contests is not fully characterized in situations where agents have private information about ability or the cost of production.

In this paper, we study repeated contests in a framework designed to capture both moral hazard (hidden effort choice) and adverse selection (privately known abilities). That is, when contestants' abilities are private information, the contestants must consider the signaling effect that exerting effort in early contests will have in future contests. Contrary to the conventional wisdom that all contestants want to appear strong to their opponents, countervailing incentives emerge in this setting and strategies are non-monotonic. These incentives can create multiple economic inefficiencies, depending on the economic context. First, overall output in repeated contests is lower relative to the single contest benchmark. In the contests described above, this reduction in output is a negative welfare result. Second, a player with low ability may beat a player with high ability in the first contest or both contests. In multi-stage tournaments, this may prevent the best contestant from winning the tournament, or even making the later rounds.

[1] Gallini and Scotchmer (2002) survey the discussion about the optimal "effective time" for the length of intellectual property rights that provides incentives for initial innovations without stifling subsequent innovation. In labor market competitions, see Ridlon and Shin (2013), Ederer (2010), and Aoyagi (2010).

We consider the simplest setting that captures the signaling incentives of repeated contests: two contestants, who have either low or high ability, competing in two successive contests. In each contest, the contestants exert effort with the goal of producing the most output. The player who does so wins a prize.[2] The amount of effort it takes to produce output depends on individual ability, which is privately known by each contestant. After the first contest, the output of each contestant is publicly observed and players can update their beliefs about their opponents' ability. Given this additional information, contestants choose a new level of effort for the second contest. The contestants choose their effort levels to maximize their total payoff over the two contests.

We show there is a unique symmetric equilibrium for this repeated contest game. To derive equilibrium strategies, we utilize the findings in the single all-pay contest with asymmetric information presented by Siegel (2014). In particular, we use his equilibrium construction to solve for the optimal strategies and expected payoffs in the second contest for any set of beliefs. From these payoffs, we show that a contestant with high ability will always prefer to have his opponent believe they have low ability. Likewise, a contestant with low ability wants to appear to have high ability. While uniqueness of equilibrium in dynamic games with signaling is not common, the incentives to misrepresent type in this setting rule out the possibility of different off-path beliefs which would be necessary to form any other equilibria. Additionally, these incentives lead to an equilibrium that has partial pooling in the first contest, i.e. there are outputs which can be produced by either low ability or high ability contestants. Low ability players who produce output in this range are *bluffing* while high ability players who do so are *sandbagging*.[3]

Bluffing is used to discourage an opponent by appearing to be strong. In our setting, this means having high ability. Avery (1998) shows that this type of behavior can emerge in a single dynamic contest where bidders submit jump bids. Additionally, in dynamic contests with hidden actions and incomplete information, contestants bluff by signal-jamming their opponents. This is found in models of labor market contests (Ederer (2010)), all-pay auctions (Ortega Reichert (2000)), and duopoly competition (Mirman et al. (1993)). Sandbagging, as described in Rosen (1986), is used to lull opponents into a false sense of security.[4] In a framework similar to ours, but with only one type of active contestant, Münster (2009) shows that the active contestant may sandbag by sitting out of the first contest in order to win the second contest more easily.

Both bluffing and sandbagging are present when bidders are allowed to send costly signals before an auction as in Hörner and Sahuguet (2007). If the bidder makes a sunk investment in the form of a jump bid before the auction, then the other bidder must match it to enter the auction. Bidders with moderate values may use a jump bid to keep other competitors out of the auction while bidders with high values may allow opponents to enter the auction freely in order to appear weak and face less competition

[2]For a general description of static (one-shot) games of this kind, see Siegel (2009).

[3]The terms *sandbag* and *bluff* are used in the literature to describe a player signaling to his opponent that he is weak when he is actually strong and strong when he is actually weak, respectively. These terms originate from the game of poker. In poker, *sandbagging* is when a player calls or does not increase the pot when he believes he has the better hand. *Bluffing* is when a player bids up the pot when he does not think he has the best hand.

[4]Additionally, sandbagging can be used to take advantage of a tournament structure as described by Kräkel (2014).

in the auction itself.

In our setting of repeated contests, both tactics are used because each contestant is concerned with how opponents of similar ability react to the outcomes of the first contest. For example, low ability contestants would be discouraged facing a strong opponent while a high ability contestant would react by increasing effort. On the other hand, a low ability contestant would be encouraged by a weak opponent and increase effort while a high ability contestant would reduce effort, thinking he could win with ease. As in Hörner and Sahuguet (2007), this leads to non-monotonicities in the equilibrium, where a contestant with low ability may beat one with high ability in either one, or both of the contests. In elimination tournaments and multi-stage auctions, designers often prefer to have the best contestants in the final round.[5] However, if the actions of the first round are publicly observable, top contestants would be worried about revealing information about their ability to their future opponents. This may cause them to lose to a lesser opponent in the first round, leading to a less competitive final round.

Lastly, we consider the effect of multiple contests on the aggregate output of the contestants. The consequence of bluffing and sandbagging is a decrease in aggregate output in the first contest when the difference between high and low ability contestants is large enough. In the setting of Münster (2009), the one active type has an incentive to sandbag, and output in the first contest is always reduced compared to a single contest benchmark. On the other hand, the effects are the opposite in Ederer (2010) before the midterm evaluation. All contestants have the incentive to bluff in order to discourage their opponents after the evaluation. This increases aggregate output before the evaluation relative to the setting where no evaluation was given.

Despite the incentives to hide true ability, partial pooling in the first contest leads to asymmetries in the second contest. These asymmetries lead to reduced aggregate output in the second contest. This is consistent with the results of Che and Gale (2003) who show that asymmetries between contestants reduces total sunk expenditures in the contest. Combining the effects of the first and second contests, we show that the reduction of output in the second contest always outweighs the potential increase in output in the first contest. Therefore the aggregate output of contestants in repeated contests is lower than the output of a single contest benchmark.[6]

The paper is organized as follows. In section 2 we introduce the model of the contest played in each stage. In section 3, we characterize the equilibrium of a single contest and discuss the payoffs to the contestants in terms of their ability and their opponent's perception of their ability. In section 4 we solve for the unique equilibrium of the successive contest. In section 5 we discuss welfare implications. Section 6 concludes.

[5]See Moldovanu and Sela (2006) and Fullerton and McAfee (1999), respectively.

[6]In dynamic contests with complete information and uncertain outcomes, contestants who fall behind in early stages will put in less effort moving forward (e.g. Harris and Vickers (1987)). Because each period is a separate contest in our model, reduced output is not due to this discouragement effect. See Konrad (2012) for a detailed survey of dynamic contests under complete information.

2 Stage Game

We first introduce the fundamentals of the contest that is played in each stage of the game of successive contests. Two contestants, player 1 and player 2, are independently endowed with ability, a^i for $i = 1, 2$. Ability can either be low, $a = a_\ell$, or high, $a = a_h$. The probability of each player having high ability is given by $Pr(a^1 = a_h) = \mu_1$ and $\Pr(a^2 = a_h) = \mu_2$. The endowment of ability is private information for each player.

Other than probability of having high ability, the two contestants are ex-ante identical. Players compete by producing output, x, which is a function of their ability and effort, e. We assume the output function takes the form $x(a, e) = a \cdot e$. The player that produces the most output receives a prize. If the two players produce the same output, then the prize is given randomly with each contestant winning with equal probability. The prize has the same value for each contestant which is normalized to one.

We define $c(e)$ to be the cost function of effort for each player, regardless of ability. This function is assumed to be twice differentiable on the non-negative reals, increasing and weakly convex, with the cost of zero effort being zero.

We normalize ability so that $a_\ell = 1$ and $a_h > 1$. Then the marginal cost of output for the high and low ability workers are $\frac{1}{a_h}c'(\frac{x}{a_h})$ and $c'(x)$ respectively.[7] The payoffs of each agent are

$$\tilde{\pi}^i(a^i, e^i, a^{-i}, e^{-i}) = \begin{cases} 1 - c(e^i), & x(a^{-i}, e^{-i}) < x(a^i, e^i) \\ 1/2 - c(e^i), & x(a^{-i}, e^{-i}) = x(a^i, e^i) \\ -c(e^i), & x(a^{-i}, e^{-i}) > x(a^i, e^i) \end{cases}$$

Given a strategy of their opponent, the expected payoffs of each contestant is equal to the probability that the contestant wins less his cost of effort. Here we abuse notation and let $x^i = x(a^i, e^i)$ for $i = 1, 2$. Then the expected payoffs for each player are

$$E[\tilde{\pi}^i(a^i, e^i)] = \Pr(x^{-i} < x(a^i, e^i)) + \frac{1}{2}\Pr(x^{-i} = x(a^i, e^i)) - c(e^i).$$

Since players know their own ability and the relationship between effort and output is deterministic, players choosing their effort level is equivalent to them choosing their output.[8] Therefore, we will write the strategies of players in terms of output to ease comparisons of players with different abilities. Additionally, it puts players' strategies in terms of what their opponents will observe. With this in mind, we write contestants' payoffs in terms of output.

$$E[\pi^i(x^i, a^i)] = \Pr(x^{-i} < x^i) + \frac{1}{2}\Pr(x^{-i} = x^i) - c(x^i/a^i), \text{ for } i = 1, 2.$$

In the following sections, we will define players' strategies in terms of output and describe the effort of players only in the context of providing intuition for the results.

[7] The convexity of the cost function amplifies abilities effect on the marginal cost of output as $\frac{\partial}{\partial x}c\left(\frac{x}{a}\right) = \frac{1}{a}c'\left(\frac{x}{a}\right)$, where $c'(\frac{x}{a_h}) \leq c'(x)$.

[8] Equivalent to the notion of private information about ability is private information about the cost of output. Additionally, if the cost of effort is linear, then this framework is equivalent to an all-pay auction where values are private information and bids are observed

In the section immediately following, players will maximize payoffs above in a single contest. In section 4, which contains the main model of successive contests, abilities for each player stay the same for both contests, and players will be maximizing the sum of payoffs for two contests, without discounting.

3 Single Contest

We first find the equilibrium strategies of players engaged in a single contest described in the previous section. This will serve two purposes when we analyze repeated contests. First, the payoffs of the contestants and the output they produce in a single contest will serve as a benchmark to better understand the strategic effects of an additional contest. Second, the equilibrium payoffs will be used to calculate continuation payoffs in the repeated contest setting. After the first of two contests, each player will believe that their opponent has high ability with some probability. For each set of these probabilities, the single contest equilibrium characterized in this section will be played in the second contest. Therefore, the expected payoffs of contestants in this section will be equal to the continuation payoffs of the second contest in the next section.

For the remainder of this section, we name our two players the strong player and the weak player, so that $i = s, w$ where $\mu_s \geq \mu_w$. This implies that, the strong player, player s, is at least as likely to have high ability as player weaker player, player w, ex-ante. However, this does not rule out the possibility of the weaker player having high ability or the stronger player having low ability, or both.

3.1 Strategies

The strategies of each player consist of output distributions for both high and low ability realizations. We define $L_i(x|\mu_i, \mu_{-i}) \equiv \Pr(x^i \leq x|a^i = a_\ell, \mu_i, \mu_{-i})$ and $H_i(x|\mu_i, \mu_{-i}) \equiv \Pr(x^i \leq x|a^i = a_h, \mu_i, \mu_{-i})$ which denote the distribution of output of player i given he has low ability and high ability respectively. Additionally, we define $F_i(x|\mu_i, \mu_{-i}) \equiv \Pr(x^i \leq x|\mu_i, \mu_{-i})$ to be the ex-ante output distribution of player i. This is also the output distribution of player i from the perspective of player $-i$. For these distributions to be consistent with the information sets of the contestants, we must have $F_i(x|\mu_i, \mu_{-i}) = (1 - \mu_i)L_i(x|\mu_i, \mu_{-i}) + \mu_i H_i(x|\mu_i, \mu_{-i})$. Additionally, we let $\ell_i(x|\mu_i, \mu_{-i})$, $h_i(x|\mu_i, \mu_{-i})$ and $f_i(x|\mu_i, \mu_{-i})$ be the densities induced from $L_i(x|\mu_i, \mu_{-i})$, $H_i(x|\mu_i, \mu_{-i})$ and $F_i(x|\mu_i, \mu_{-i})$.[9] For simplicity, we suppress the probabilities, (μ_i, μ_{-i}), from the notation of the output distributions for the remainder of this section.

We denote support of the strategies of each type of player by $X^i_\ell \equiv \{x : \ell_i(x) > 0\}$ and $X^i_h \equiv \{x : h_i(x) > 0\}$. For a given expected output distribution of their opponent, the best response set for a given contestant and given ability is defined as

$$BR_i(a^i) \equiv \{x : E[\pi^i(x^i, a^i)] \geq E[\pi^i(\tilde{x}^i, a^i)], \forall \tilde{x}^i \geq 0\}.$$

[9]Here we use the extended definition of density using Dirac-delta functions where necessary to properly define these densities when their corresponding distributions have mass points.

An equilibrium is a set of output distributions, $(L_s(x), H_s(x), L_w(x), H_w(x))$, such that $X^i_\ell \subseteq BR_i(a_\ell)$, and $X^i_h \subseteq BR_i(a_h)$ for $i = s, w$. The general properties of an equilibrium are outlined in the following lemma; the proof is in the appendix.

Lemma 3.1. *In any equilibrium, players' distributions of output, $H_s(x), L_s(x), H_w(x)$, and $L_w(x)$, are continuous on $(0, x^*)$, where $x^* \equiv \sup\{BR_s(a_\ell) \cup BR_s(a_h)\} = \sup\{BR_w(a_\ell) \cup BR_w(a_h)\}$ and $\inf\{BR_s(a_\ell) \cup BR_s(a_h)\} = \inf\{BR_w(a_\ell) \cup BR_w(a_h)\} = 0$.*

From these properties, it must be that for both the strong and weak contestants, the combined best response sets of the low ability and high ability type must be an interval. Moreover, this interval for each contestant is the same. Intuitively, if the supremum of the interval was smaller for one of the contestants, then the other contestant would be wasting effort by sometimes producing more output than would ever be necessary to win the contest. Moreover, if there are gaps of positive measure in this combined interval for either contestant, then the opponent would have no incentive to produce output in the interior of the gap. However, this leads to a gap in the best response intervals for both players, but this cannot happen. By this argument, which is formalized in the proof of Lemma 3.1, the infimum of the best response interval of each player must be 0. Lastly, if either player played a positive output with positive probability, this implies that their opponent must have a gap in their combined best response interval behind this output. We argued that this gap cannot exist.

While the fundamentals of this model are somewhat different to those studied by Siegel (2014), the properties of the equilibrium strategies above are the same. Moreover, he shows that when types are independently drawn and value of winning the contest increases in type, then there is a unique equilibrium which must be monotonic, i.e., for both contestants, all actions of the high type are at least as high as all actions of the low type. These properties hold in our setting where types are abilities and bids are outputs.[10] This implies that there is a unique equilibrium that is monotonic, so that for $i = s, w$ and any $x \in BR_i(a_h)$ and $x' \in BR_i(a_\ell)$ it must be that $x' < x$. This fact, combined with the previous lemma implies that $x^*_i \equiv \sup\{BR_i(a_\ell)\} = \inf\{BR_i(a_h)\}$, for $i = s, w$.

Proposition 3.2 (Unique Equilibrium - Single Contest)**.** *There is a unique equilibrium, $(L^*_s(x), H^*_s(x), L^*_w(x), H^*_w(x))$, where $\overline{X^\ell_i} = BR_i(a_\ell) = [0, x^*_i]$, $\overline{X^h_i} = BR_i(a_h) = [x^*_i, x^*]$ for $i = s, w$ and $0 \le x^*_s \le x^*_w \le x^*$. These output distributions are given by*

$$L^*_s(x) = \begin{cases} \frac{c(x)}{1-\mu_s}, & 0 \le x \le x^*_s \\ 1, & x^*_s \le x \le x^* \end{cases} \qquad H^*_s(x) = \begin{cases} 0, & 0 \le x \le x^*_s \\ \frac{c(x)}{\mu_s} - \frac{c(x^*_s)}{\mu_s}, & x^*_s \le x \le x^*_w \\ 1 + \frac{c(x/a_h)}{\mu_w} - \frac{c(x^*/a_h)}{\mu_w}, & x^*_w \le x \le x^* \end{cases}$$

$$L^*_w(x) = \begin{cases} \frac{c(x)}{1-\mu_w} + L^*_w(0), & 0 \le x \le x^*_s \\ 1 + \frac{c(x/a_h)}{1-\mu_w} - \frac{c(x^*_w/a_h)}{1-\mu_w}, & x^*_s \le x \le x^*_w \\ 1, & x^*_w \le x \le x^* \end{cases} \qquad H^*_w(x) = \begin{cases} 0, & 0 \le x \le x^*_w \\ 1 + \frac{c(x)}{\mu_w} - \frac{c(x^*)}{\mu_w}, & x^*_w \le x \le x^* \end{cases}$$

[10] While, in this section, we borrow heavily from the properties of Siegel (2014) and follow his construction to characterize the unique equilibrium, the setting is slightly different and therefore we cannot directly use his results. His contestants differ on value of the prize rather than ability and the cost of output, which is a simple bid, is linear. Monotonicity holds in the current model as a player's marginal cost for a given output is ranked by type, where his contestants marginal value is ranked by type for all bids. A simple transformation makes these two settings isomorphic.

where $x_s^* = c^{-1}(1-\mu_s)$, $x_w^* = c^{-1}(1-\mu_w)$, *and* $x^* = a_h c^{-1}\left(\mu_w + c\left(\frac{c^{-1}(1-\mu_w)}{a_h}\right)\right)$, *and* $L_w^*(0) = \frac{1}{1-\mu_w}\left[\mu_s - \mu_w - \left[c\left(\frac{c^{-1}(1-\mu_w)}{a_h}\right) - c\left(\frac{c^{-1}(1-\mu_s)}{a_h}\right)\right]\right]$.

Here we highlight the important details of the construction of the equilibrium; see the appendix for the technical details.

First from Lemma 3.1, the combined best response sets for the strong player and the weak player are the same, which we denote by the interval, $[0, x^*]$. Second, since the equilibrium is monotonic, the best response sets of each ability are disjoint for each player, with the set for high ability ranging over larger outputs than the set for low ability. Third, since the strong player is more likely to be high ability, the length of the best response set of high ability for is longer for the strong player. We depict the basic structure of these best response sets in Figure 1.

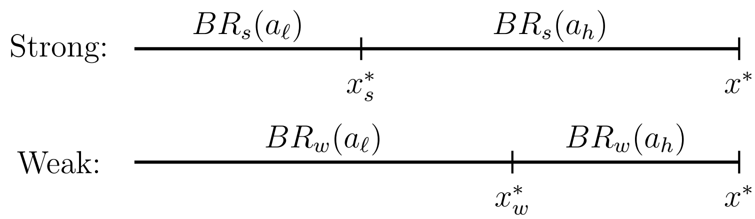

Figure 1: Representation of best response sets of the strong and weak players.

To characterize the output distributions of each contestant for each ability level, we first find the expected output distributions that each contestant must face to be indifferent between output levels when each best response set. We start from x^* and work backwards toward zero. We will be able to pin down the value of x^* and subsequently x_w^* and x_s^*, using that fact that only one player can have a mass point at $x = 0$, $F_i(0) > 0$, and for the other $F_{-i}(0) = 0$ and $F_{-i}(x) > 0$ for $x > 0$. Therefore the expected output distribution that hits zero first will pin down the x^*.

For output levels between x^* and x_w^*, the high ability type of each contestant must be indifferent. This means that the marginal benefit of increasing output must equal the marginal cost, i.e. $f_i(x) = c'(x/a_h)$ for $x \in (x_w^*, x^*)$ and $i = s, w$. This implies that the output distributions for this range of output are the same for both contestants. For output between x_w^* and x_s^*, $f_s(x)$ must equal the marginal cost of the weak contestant and $f_w(x)$ is equal to the marginal cost of the strong contestant. Since this range is a best response of the low ability type of the weak player and the high ability type of the strong player, this implies that $f_s(x) = c'(x/a_\ell) = c'(x)$ and $f_w(x) = c'(x/a_h)$. Lastly, for the output range of x_s^* to $x = 0$, the low ability type of each player must be indifferent, and therefore $f_i(x) = c'(x)$.

Given the densities for all levels of output, we can compute the ex-ante expected output distributions for each player given the condition that $F_i(0) = 0$ for one contestant. This must be the strong contestant as $f_s(x) \geq f_w(x)$ for all output levels between 0 and x^*. Intuitively, it is the weaker contestant that has a mass point at zero, i.e. if this player draws low ability, they may not put any effort into the contest.

The expected output distribution of the weak and strong players are

$$F_s^*(x) = \begin{cases} c(x), & 0 \leq x \leq x_w^* \\ 1 - \mu_w - c\left(\frac{c^{-1}(1-\mu_w)}{a_h}\right) + c\left(\frac{x}{a_h}\right), & x_w^* \leq x \leq x^* \end{cases}$$

$$F_w^*(x) = \begin{cases} F_w(0) + c(x), & 0 \leq x \leq x_s^* \\ 1 - \mu_w - c\left(\frac{c^{-1}(1-\mu_w)}{a_h}\right) + c\left(\frac{x}{a_h}\right), & x_s^* \leq x \leq x^* \end{cases}$$

An example of these distributions with $a_h = 2$ and $c(e) = \frac{1}{2}e^2$ is depicted below.

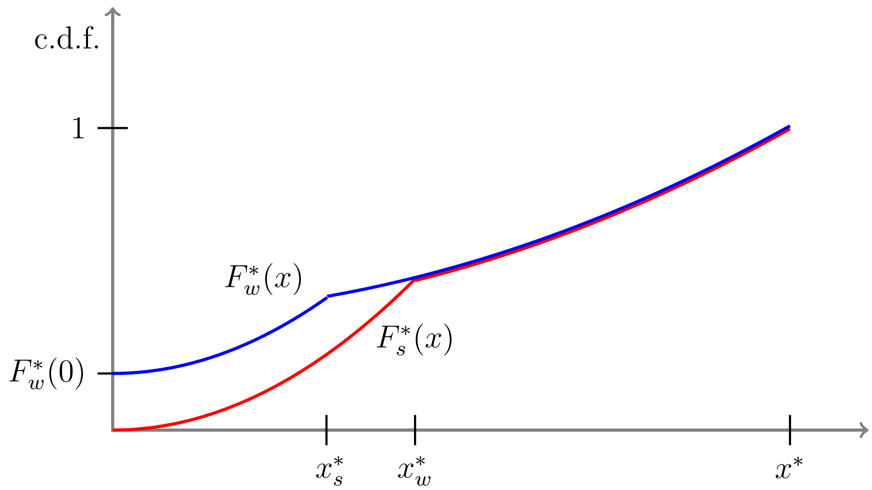

Figure 2: Expected output distributions of contestants in a single contest.

Because the best response sets of low and high ability types are disjoint for each contestant, we can recover the output distributions of both high and low ability types of both contestants.

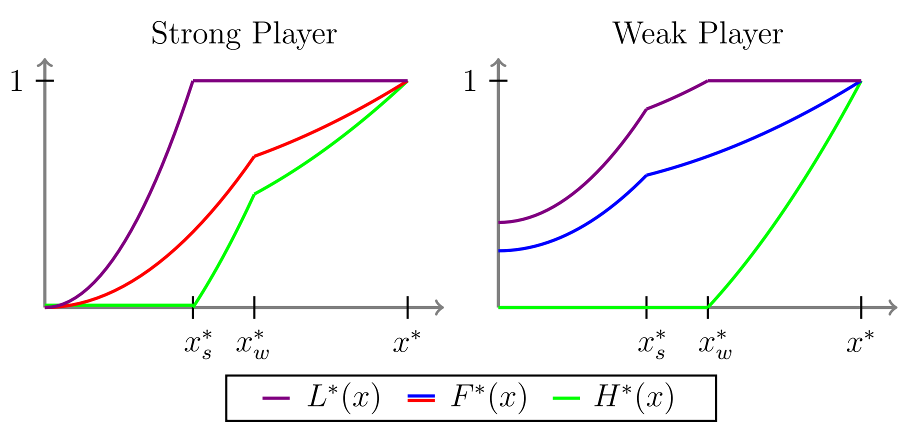

Figure 3: Output distributions for high and low ability type of the strong and weak player in a single contest. ($a_h = 2$ and $c(e) = \frac{1}{2}e^2$)

3.2 Payoffs

From the equilibrium of the single contest, our main objects of interest are the payoffs of the contestants. Given the uniqueness of this equilibrium for any pair of probabilities (μ_s, μ_w), these payoffs will equal the expected payoffs the contestants expect to recieve in the second contest, given the beliefs that result from the first contest. We denote the payoffs, which are functions only of the contestant's ability and the ex-ante probabilities of each contestant being high ability as $v_i(\mu_i, \mu_{-i}, a^i) = E[\pi^i(\hat{x}^i, a^i)]$ where $\hat{x}^i \in BR_i(a^i)$.

Corollary 3.3. *The ex-interim expected payoff of each contestant is*

$$\begin{aligned} v_s(\mu_s, \mu_w, a_h) = v_w(\mu_w, \mu_s, a_h) = 1 - \mu_w - c\left(\frac{c^{-1}(1-\mu_w)}{a_h}\right) \\ v_s(\mu_s, \mu_w, a_\ell) = \mu_s - \mu_w - \left[c\left(\frac{c^{-1}(1-\mu_w)}{a_h}\right) - c\left(\frac{c^{-1}(1-\mu_s)}{a_h}\right)\right] \\ v_w(\mu_w, \mu_s, a_\ell) = 0. \end{aligned}$$

Proof. Each type of each contestant is indifferent between all outputs in their best response set. In particular, because x^* is in the best response set of high ability contestants, their expected payoffs are equal to the value of winning less the cost of producing output x^*, since, if they produce x^*, they will win with certainty.

$$v_s(\mu_s, \mu_w, a_h) = v_w(\mu_w, \mu_s, a_h) = 1 - c(x^*/a_h) = 1 - \mu_w - c\left(\frac{c^{-1}(1-\mu_w)}{a_h}\right)$$

Similarly, since $x = 0$ is in the best response set of a low ability contestant, the expected payoffs of low ability contestants is probability they win, given they exert no effort. This is the probability that your opponent puts in no effort.[11]

$$v_s(\mu_s, \mu_w, a_\ell) = (1-\mu_w)L_w(0) = \mu_s - \mu_w - \left[c\left(\frac{c^{-1}(1-\mu_w)}{a_h}\right) - c\left(\frac{c^{-1}(1-\mu_s)}{a_h}\right)\right]$$

$$v_w(\mu_w, \mu_s, a_\ell) = (1-\mu_s)L_s(0) = 0$$

□

For both the strong and weak contestants who are high ability, the expected payoff is entirely determined by the value of x^*. This value is pinned down by the ex-ante expected output distribution of the stronger contestant which is constructed by making the weaker player indifferent. Therefore, x^* is a determined entirely by μ_w, the probability that the weaker player has high ability. Intuitively, high ability players are confident they can win, but the overall competitiveness of the contest will determine how much effort they need to exert to do so. This payoff decreases as μ_w increases, implying that increased competition will increase the effort of high ability players, decreasing their expected payoff.

[11] Here we technically are assuming the contestant wins all ties at zero, but if the agent exerts a tiny amount of effort and we let that effort shrink to zero, then this is the payoff of the agent in the limit. Since the payoffs are continuous, these limits must be the payoffs of the low ability contestants.

For contestants that are low ability, expected payoffs are determined by how often they can freely when a contest. The strong contestant will exert no effort with probability zero, while the weak contestant will exert no effort with a probability that increases with the strength of their opponent. Intuitively, a low ability player becomes discouraged when he believes that his opponent is likely to have high ability. Therefore, the low ability type of the weaker contestant will never when a contest when they exert no effort leading to an expected payoff of zero. The stronger contestant who has low ability will have positive expected payoffs which increase with the contestant's relative strength.

The comparative statics of these payoffs are important for the analysis of the two contest model as the payoffs in the second contest are the same as in the single contest given the strength of each player. The contestants can effect their perceived strength in this second contest through their choice of output in the first contest. For a contestant with high ability, payoffs decrease when the contest appears more competitive. Therefore, they may prefer to look weak entering the second contest in order to reduce the perceived level of competition. On the other hand, the payoffs of low ability contestants increase when they appear strong to their opponent. These countervailing incentives, which will be formalized in the following section, are a significant strategic force in the first contest of the two contest model.

3.3 Output

The second outcome of interest in the single contest is the expected total output of the contestants. We will use this output as a benchmark to compare with per-period output in the repeated contest model. In order to have a closed form solution for expected output, we impose a parametric structure to the cost function.

Corollary 3.4. *If we let $c(e) = ke^{\alpha}$, with $k > 0$ and $\alpha \geq 1$,*[12] *then $E[x_s(\mu_s, \mu_w, a^s) + x_w(\mu_w, \mu_s, a^w)]$, the ex-ante expected aggregate output is*

$$\left(\frac{\alpha}{\alpha+1}\right)\left(\frac{1}{k}\right)^{1/\alpha}\left[\left(1-\frac{1}{a_h^{\alpha}}\right)\left((1-\mu_w)^{\frac{\alpha+1}{\alpha}}+(1-\mu_s)^{\frac{\alpha+1}{\alpha}}\right)+2a_h\left(\mu_w+\frac{1-\mu_w}{a_h^{\alpha}}\right)^{\frac{\alpha+1}{\alpha}}\right].$$

A significant determinant of this total output is μ_w, or the probability the weaker contestant has high ability. Additionally, while an increase in μ_w will increase output, an increase in μ_s will have the opposite effect.

Corollary 3.5. *For a fixed value of μ_w, an increase in μ_s leads to a reduction in expected aggregate output.*

Proof.

$$\frac{\partial}{\partial \mu_s} E[x_s(\mu_s, \mu_w, a^s) + x_w(\mu_w, \mu_s, a^w)] = -\left(\frac{1}{k}\right)^{1/\alpha}\left(1-\frac{1}{a_h^{\alpha}}\right)(1-\mu_s)^{\frac{1}{\alpha}} < 0.$$

□

[12] These are implied by the assumptions of a cost function that is strictly increasing and weakly convex

This reflects the decrease in the overall competition of the contest when one contestant is stronger than the other. Therefore the overall competitiveness of a contest, in the sense of expected output produced, is effected both by the absolute strength of the two contestants and their strengths relative to each other. Intuitively, if a very strong contestant, $\mu_s \approx 1$, and a very weak contestant, $\mu_w \approx 0$, compete against each other, the weak person would likely put in little or no effort, thinking that winning is very unlikely. Additionally, the strong contestant would know the weak contestant is putting in low effort and reduce their effort accordingly.

4 Repeated Contests

We now turn to the main model of the paper, a game of two repeated contests where agents are privately informed about their ability. We assume the contestants are symmetric, ex-ante, and they are equally likely to have low or high ability. The contestants maximize the sum of expected payoffs in each contest, and their abilities do not change after the initial draw of types.

After realizing their respective abilities, contestants play the first contest. Once the first contest ends, the outputs of each contestant become public information. Contestants use this information to update their beliefs about their opponent's ability prior to competing in the second contest. Because these outputs are commonly observed, first order beliefs are sufficient for characterizing optimal strategies. In particular, beliefs which are consistent with the first period equilibrium strategy will have the same effect on the strategies of the second contest as the ex-ante probabilities of being high ability have in the single contest model.

Given a strategy of player $-i$, we denote expected payoffs of player i over the two contests as

$$\begin{aligned}
E[\pi^i(x_1^i, x_2^i, a^i)] &= E[\pi_1^i(x_1^i, a^i)] + E[\pi_2^i(x_2^i, a^i)|\mu(x_1^i)] \\
&= \Pr(x_1^{-i} < x_1^i) + \frac{1}{2}\Pr(x_1^{-i} = x_1^i) - c\left(x_1^i/a^i\right) \\
&+ \Pr(x_2^{-i} < x_2^i|\mu(x_1^i)) + \frac{1}{2}\Pr(x_2^{-i} = x_2^i|\mu(x_1^i)) - c(x_2^i/a^i) \text{ for } i = 1, 2.
\end{aligned}$$

4.1 Strategies

For each player $i = 1, 2$, we let $L_1^i(x)$ and $H_1^i(x)$ denote the first period output distributions of a contestant with low ability and high ability respectively. Then the ex-ante expected output distribution is $F_1^i(x_1) = \frac{1}{2}L_1^i(x_1) + \frac{1}{2}H_1^i(x_1)$, for $i = 1, 2$. Additionally, we denote f_1^i, ℓ_1^i and h_1^i as the densities that are induced from the distribution functions F_1^i, L_1^i and H_1^i.[13] Lastly, define $X_1^{h,i} = \{x|h_1^i(x) > 0\}$ and $X_1^{\ell,i} = \{x|\ell_1^i(x) > 0\}$. Since contestant's are symmetric, we will restrict attention to equilibria that are symmetric.

A set of output distributions $\{H_1^i(x_1), L_1^i(x_1), H_2^i(x_2|\mu_i, \mu_{-i}), L_2^i(x_2|\mu_i, \mu_{-i})$ for $i = 1, 2\}$ form a symmetric perfect Bayesian equilibrium (SPBE) for two successive contests if

[13] Again we are using the extended definition of density using Dirac-delta functions where necessary.

1. strategies are symmetric: $H_1^1(x) = H_1^2(x)$, $L_1^1(x) = L_1^2(x)$, $H_2^1(x|\mu_1, \mu_2) = H_2^2(x|\mu_2, \mu_1)$, and $L_2^1(x|\mu_1, \mu_2) = L_2^2(x|\mu_2, \mu_1)$,

2. contestants play the unique Bayes Nash equilibrium in the second contest: for $i = 1, 2$ and for every (μ_i, μ_{-i}),

$$(H_2^i(x|\mu_i, \mu_{-i}), L_2^i(x|\mu_i, \mu_{-i})) = \begin{cases} (H_w^*(x|\mu_i, \mu_{-i}), L_w^*(x|\mu_i, \mu_{-i})), & \text{if } \mu_i \leq \mu_{-i} \\ (H_s^*(x|\mu_i, \mu_{-i}), L_s^*(x|\mu_i, \mu_{-i})), & \text{if } \mu_i > \mu_{-i} \end{cases}$$

3. players update beliefs according to Bayes rule when feasible:[14]

$$\mu_i = \mu(x_1^i) = \frac{h_1(x_1^i)}{h_1(x_1^i) + \ell_1(x_1^i)}, \text{ for } i = 1, 2, \text{ and}$$

4. given (2) and (3), contestants always choose an optimal output in the first contest: for $i = 1, 2$, for every $x_1^i \in X_1^{\ell,i}$ player i chooses an

$$x_1^i \in \arg\max_{x^i} E[\pi(x_1^i, x_2^i(\mu(x_1^i), \mu(x_1^{-i}), a_\ell), a_\ell)] \equiv BR_i(a_\ell),$$

and for every $x_1^i \in X_1^{h,i}$ player i chooses an

$$x_1^i \in \arg\max_{x^i} E[\pi(x_1^i, x_2^i(\mu(x_1^i), \mu(x_1^{-i}), a_h), a_h)] \equiv BR_i(a_h).$$

In the single contest section, we found the unique strategies that satisfy condition (2). To find the strategies of each type of player in the first period that satisfy (4), we first examine how output in the first contest affects expected payoffs in the second contest. From condition (2), for a given set of beliefs that arise from outputs in the first period, the expected payoffs for each player are $v_i(\mu(x_1^i), \mu(x_1^{-i}), a^i)$. Therefore the payoffs to player i for the two contests can be written in terms of output in the first contest.

$$E[\pi^i(x_1^i, \hat{x}_2^i(\mu(x_1^i), \mu(x_1^{-i}), a^i), a^i)] = E[\pi_1^i(x_1^i, a^i)] + E[v_i(\mu(x_1^i), \mu(x_1^{-i}), a^i)]$$

Previous analysis showed that for a given belief of the opponent, a high ability contestant will have higher expected payoffs if his opponent believes he is low type with high probability. Furthermore, it showed that a low ability contestant will have higher expected payoffs if his opponent believes he is high type with high probability. The following proposition shows that in expectation, high ability players always prefer to look weaker entering the second contest, while low ability players always prefer to look stronger. The proof of this proposition and other results of this section are relegated to the appendix.

Proposition 4.1 (Countervailing Incentives)**.** *For all $\mu_i \in (0, 1)$, expected payoffs in the second contest decrease for high ability players as μ_i increases, $\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_h)] < 0$, and increase with μ_i for low ability players, $\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_\ell)] > 0$.*

[14] Using the extended definition of density allows agents to update their beliefs even when they see their opponent produce an output where the distribution has a mass point. For example, if the H_1 has a mass point at x, while L_1 does not, this definition implies $\mu(x_1) = 1$.

In particular, the marginal effect of beliefs on payoffs in the second contest is given by

$$\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_h)] = d(\mu_i)(F_{\mu_{-i}}(\mu_i) - 1) < 0 \text{ and}$$

$$\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_\ell)] = d(\mu_i) F_{\mu_{-i}}(\mu_i) > 0,$$

$$\text{where } d(\mu_i) \equiv \left[1 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(1-\mu_i)}{a_h}\right)\right] \text{ and } F_{\mu_{-i}}(\mu_i) = \Pr(\mu_{-i} \leq \mu_i).$$

The convexity of the cost of effort implies that $d(\mu_i) \in \left[\frac{a_h - 1}{a_h}, 1\right)$ for all μ_i, which guarantees that these incentives are strict.

Differences in marginal cost of effort for the first contest, and the incentives stemming from the second contest combine to require that the belief function, $\mu(x)$, increases in first period output, in equilibrium. If it does not, a higher output would result in a lower belief about the ability of the contestant. From condition (3), this implies that the higher output must be in the $BR(a_\ell)$ and the lower output will be in $BR(a_h)$. However, if the lower output $\in BR(a_h)$, then the low ability player must strictly prefer the lower output to the higher output, as the marginal cost of this contestant for the high output is larger and the expected payoffs in the second contest would be higher for the lower output.

Corollary 4.2 (Monotonic Beliefs)**.** *In every SPBE, $\mu(x)$ is weakly increasing in x for all $x \in X_1 = X_1^h \cup X_1^\ell$.*

In addition to restricting the belief function on the equilibrium path, the countervailing incentives also eliminate multiplicity of equilibria. Therefore, in this setting, there is a unique symmetric equilibrium even without additional equilibrium refinements.

Theorem 4.3 (Uniqueness of Equilibrium)**.** *There is a unique symmetric perfect Bayes Nash equilibrium $\{(L_1^*(x_1), L_2^*(x_2|\mu_i, \mu_{-i})), (H_1^*(x_1), H_2^*(x_2|\mu_i, \mu_{-i}))\}$.*

The following lemmas show that equilibrium strategies in the first contest have no atoms and there are no gaps in the best response sets.

Lemma 4.4. *There is no output that is played with positive probability and* $\Pr(win|x) = F_1(x)$ *is continuous.*

Lemma 4.5. *$BR(a_\ell)$ and $BR(a_h)$ are intervals where $0 = x_{\ell,*} \leq x_{h,*} < x_\ell^* \leq x_h^*$ and we define $x_{\ell,*} = \inf\{BR(a_\ell)\}, x_\ell^* = \sup\{BR(a_\ell)\}, x_{h,*} = \inf\{BR(a_h)\}$ and $x_h^* = \sup\{BR(a_h)\}$.*

Lemma 4.4 implies that first period payoffs are continuous in output. Furthermore, there can be no gaps in the best responses of each type of contestant. In other games with signaling, gaps may exists when off path beliefs prevent players from choosing actions. However, any pathological belief will benefit at least one type of contestant due to their countervailing incentives. Therefore, if there were gaps in best response sets, and therefore outputs where density is zero for contestants of both high and low ability, then one of the player types would benefit from deviating to an output in the gap.

Additionally, the countervailing incentives imply that $BR(a_\ell) \cap BR(a_h)$, the intersection of the best response sets of the low and high ability players, is non trivial. In contrast to the equilibrium properties of a single contest, this overlap shows that there are outputs that both high and low ability contestants could optimally choose.

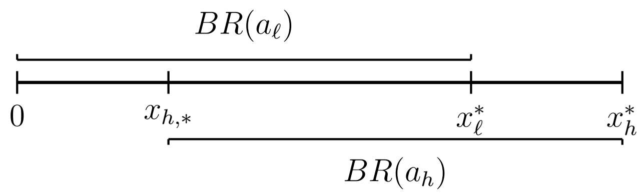

Figure 4: Representation best response sets of the high ability and low ability players in the first contest.

The lemmas above show that there are three distinct intervals in each equilibrium. These intervals are partitioned by the best response sets of the high and low ability players. The first is the set of outputs where only low ability players are optimizing: $[0, x_{h,*}) = \{BR(a_\ell) \setminus BR(a_h)\}$. Next is the set of outputs where both high and low ability players are optimizing $[x_{h,*}, x_\ell^*] = \{BR(a_\ell) \cap BR(a_h)\}$. Lastly is the set of outputs where only high ability players are optimizing: $(x_\ell^*, x_h^*] = \{BR(a_h) \setminus BR(a_\ell)\}$. For each output where $x \in BR(a_\ell)$, the low ability player's first order condition must hold and likewise for each $x \in BR(a_h)$ the high ability player's first order condition must hold.

Continuous output distribution functions and cost functions along with indifference over best response sets imply that the belief function must also be continuous. Therefore, on the three intervals, the belief function must be 0, weakly increasing, and 1 respectively.

Corollary 4.6. *The belief function and the distribution functions of output are continuous in output on $[0, x_h^*]$. Additionally, the belief function is given by $\mu(x) = 0$ for all $x \in [0, x_{h,*}]$, $\mu(x) = 1$ for all $x \in [x_\ell^*, x_h^*]$ and is weakly increasing on $(x_{h,*}, x_\ell^*)$.*

Conditions for x being in $BR(a_h)$ and $BR(a_\ell)$ are

$$BR(a_h): \; F_1^*(x) + E[v_i(\mu(x), \mu(x^{-i}), a_h)] - c\left(\frac{x}{a_h}\right) = k_h$$

$$BR(a_\ell): \; F_1^*(x) + E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] - c(x) = k_\ell = 0$$

- For the range of $0 \le x < x_{h,*}$ we have that $E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] = 0$ as $\mu(x) = 0$. Therefore we have that $F_1^*(x) = c(x)$ for all $x \in [0, x_{h,*}]$.
- For the range $x_\ell^* < x \le x_h^*$, $E[v_i(\mu(x), \mu(x^{-i}), a_h)] = E_{x_j}[v_i(1, \mu(x_j), a_h)] \equiv v_h$. Then we have $F_1^*(x) + v_h = c(x/a_h) + k_h$, for all $x \in [x_\ell^*, x_h^*]$.

- For the range $x_{h,*} \leq x \leq x^*_\ell$, for all $x \in \{X^\ell_1 \cup X^h_1\}$ we know $x \in \{X^\ell_1 \cap X^h_1\}$. Therefore, both low and high ability players are indifferent between all outputs in this range. Because the marginal cost of the low ability player is always more than the marginal cost of the high ability player, this can only be true if increasing output benefits the low ability player more than the high ability player. This indifference condition determines the belief function over this interval and as the difference in marginal benefits of increasing output for the high ability and low ability players must equal the difference in marginal costs that they face today.

  $$\mu'(x)d(\mu(x)) = c'(x) - \frac{1}{a_h}c'(x/a_h)$$

  Combining the condition on the belief function with the best response conditions of both the high and low ability contestants gives us the a condition on the density function over this interval.

  $$f^*_1(x) = \frac{\partial}{\partial x}c(x)(1 - F^*_1(x)) + \frac{\partial}{\partial x}c\left(\frac{x}{a_h}\right)F^*_1(x) \quad (\dagger)$$

Given $f_1(x)$ on $[0, x^*]$, the endpoints $x_{h,*}$, x^*_ℓ, and x^*_h can be solved for using the following conditions, $\mu(x^*_\ell) = 1$, $L_1(x^*_\ell) = 1$, continuity of F_1 at x^*_ℓ and $F_1(x^*_h) = 1$. Additionally, the equilibrium strategies of high ability and low ability players are determined by $f_1(x)$ and $\mu(x)$.

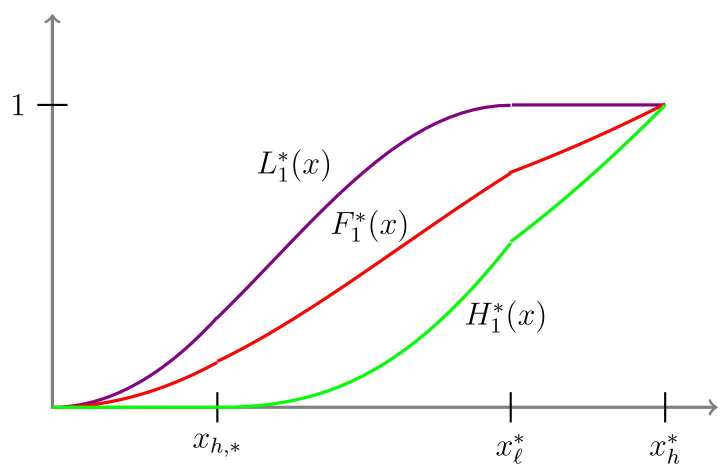

Figure 5: Distribution of strategies in the first of two successive contests.
($c(e) = \frac{1}{2}e^2$, $a_h = 2$)

4.2 Discussion

Observed output in the first contest from each contestant will land in one of three intervals. If the output is between 0 and $x_{h,*}$, then the player is revealed to have low ability, while output between x^*_ℓ and x^*_h must have been produced by a player with high ability. Output between these ranges may have been produced by a player with either high or low ability.

Low ability contestants either choose a low effort which produces an output that reveals their ability to their opponent, or they decide to *bluff* by choosing a higher effort which produces an output that may have been produced by a high ability contestant. This higher effort level will provide a lower expected payoff in the first contest as the additional cost of effort will exceed the benefit of increasing the probability of winning this contest. These contestants are willing to put in the additional effort to have a stronger position entering the second contest, as they benefit from appearing to have high ability. Therefore ex-interim expected payoffs for a low ability player are negative in the first contest and positive in the second contest.

High ability contestants either choose a high effort which produces an output that reveals their ability to their opponent, or they decide to *sandbag* by choosing a lower effort which produces an output that may have been produced by a low ability player. Since effort is less costly to high ability players, the lower cost of reduced effort will not offset the lower winning probability of the first contest. These players are willing to produce the lower output in the first contest as they benefit from appearing to have a low ability entering the second contest. The per period expected payoffs are depicted in Figure 6.

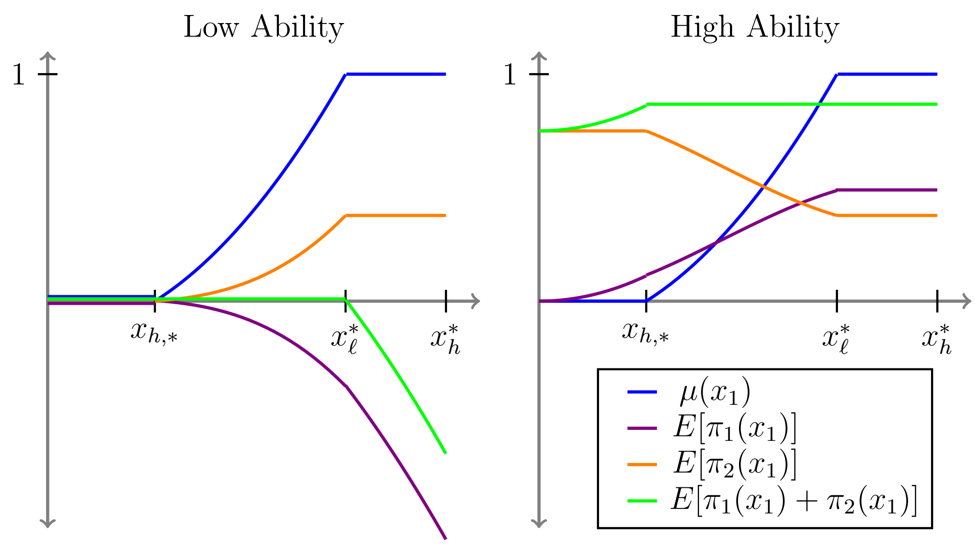

Figure 6: Beliefs and expected payoffs as a function of first period output ($c(e) = \frac{1}{2}e^2$ and $a_h = 2$).

Strategic incentives will reduce the expected output of high ability players and increase the expected output of low ability players in the first contest as compared to the expected output each type of player in a single contest. These distortions in effort in the first contest effect the aggregate output and payoffs as well as the potential outcomes of each contest.

5 Welfare

We now analyze the welfare effects of the incentives in the first round of players to hide their ability and the information released between contests. We compare the outcomes, expected payoffs and the expected output of the contestants in the repeated contest model to a benchmark where players compete with the same payoff structure but the possibility of signaling private information through actions is suppressed.

The benchmark can be thought of in two different ways. First, there are two separate contests as before, but the output (and winner) is not revealed after the first contest. Winners of both contests are revealed after the second contest. Second, you can think of it as one longer contest, where either the cost function is stretched by factor of two, or cost is a function of the intensity of effort of the contestant over the two periods. In the later case, contestants would choose an intensity level that they would maintain over the two periods.

For the benchmark, we can take the strategies from the single contest analysis. To compare this model to the repeated contest model, we assume the contestants are symmetric ex-ante, i.e. initial beliefs are taken to be $\mu_s = \mu_w = \frac{1}{2}$. Additionally, to compute expected aggregate output, we assume that the cost function of effort is given by $c(e) = ke^{\alpha}$. The distribution function of output for each contestant is

$$F(x) = \begin{cases} kx^{\alpha}, & 0 \leq x \leq \left(\frac{1}{2k}\right)^{1/\alpha} \\ \frac{1}{2A} + \frac{k}{a_h^{\alpha}} x^{\alpha}, & \left(\frac{1}{2k}\right)^{1/\alpha} \leq x \leq \left(\frac{a_h^{\alpha}+1}{2k}\right)^{1/\alpha} \end{cases}$$

5.1 Outcomes

In the equilibrium for two successive contests, overlapping best response sets give a low ability player a positive probability beating a high ability player in the first contest. Additionally, if the low ability player enters the second contest in a stronger position, which is always the case when they win the first contest, they may also win the second contest.

Corollary 5.1 (Surprise Victories)**.** *A low ability player has a positive probability of winning each contest, even if they are competing against a high ability player.*

In contrast to this, in the benchmark model, the best response sets for high and low ability players are disjoint, implying that a high ability player will always win a contest against a low ability player.

5.1.1 Application to Multi-Stage Tournaments

This fact is used to motivate a multi-stage tournament by both Moldovanu and Sela (2006) and Fullerton and McAfee (1999). However, as shown in Ye (2007), efficient entry into later rounds can not be guaranteed when contestants have private information that is preserved between rounds.

Our results show that when private information about ability is preserved and contestants can learn about future opponents through their past output, then the best player may not win the tournament, and in fact, may not make it to the final round. To see this connection, consider a four player, two stage tournament where the payoffs of each stage is identical to each contest in the current model. Output from the first stage is observed by all four players before the second stage. Because these outputs are sufficient to characterize the second round strategies, the countervailing incentives in the first round will be consistent with the current model despite the fact that a first round winner will be playing a different opponent in the second round. The difference from the current model stems from the fact that the loser of the first stage does not compete in the second stage. This would alleviate some of the distortion as high ability players would now be more motivated to win the first stage, but given the fact that they make the final round, they would still prefer to appear weak. This would leave the possibility of high ability players sandbagging in the first round and therefore not making the final round.

5.2 Payoffs

Theorem 5.2 (Increased Aggregate Payoffs)**.** *The expected payoff for the low ability player is the same per contest as the single contest benchmark, while the high ability player receives a higher expected payoff per contest.*

Proof. Equilibrium payoffs of a low ability player in the two contest model are 0. This is equal to the expected payoff in the single contest benchmark.

Payoffs of the high type in the benchmark game are given by $\frac{1}{2A}$. Payoffs for the low type are zero. If no information is revealed, then over two periods (cost functions are stretched by 2), the expected payoffs for a high ability player are $\frac{1}{A}$. If we compare this to the two period payoff of the high type in successive contests where information is revealed, then we see that it is higher as

$$k_h - \frac{1}{A} = \frac{1}{2A} + \frac{(2A-1)(1-a_h^\alpha e^{-1/A})}{2Aa_h^\alpha(1-e^{-1/A})} = \frac{1}{2A}\left(1 - \frac{(2A-1)(e^{-1/A} - 1/a_h^\alpha)}{1-e^{-1/A}}\right) > 0$$

Therefore, two period payoffs of the high ability player are higher than one long contest. □

High ability players benefit from the compression of potential outputs that arise in the first contest from players hiding information about their ability. Therefore in successive contests, we expect to see lower overall output and a higher reward for players with higher ability who are benefiting from the reduction in effort levels.

5.3 Output

Theorem 5.3 (Reduced Aggregate Output)**.** *Ex-ante expected aggregate effort of the players in each of the two contests is less than in the single contest.*

Proof. Ex-ante payoffs for the players are $\frac{k_h}{2}$ in the two period game, and $\frac{1}{2A}$ in the benchmark. Since the players are symmetric, then ex-ante, each will win each game

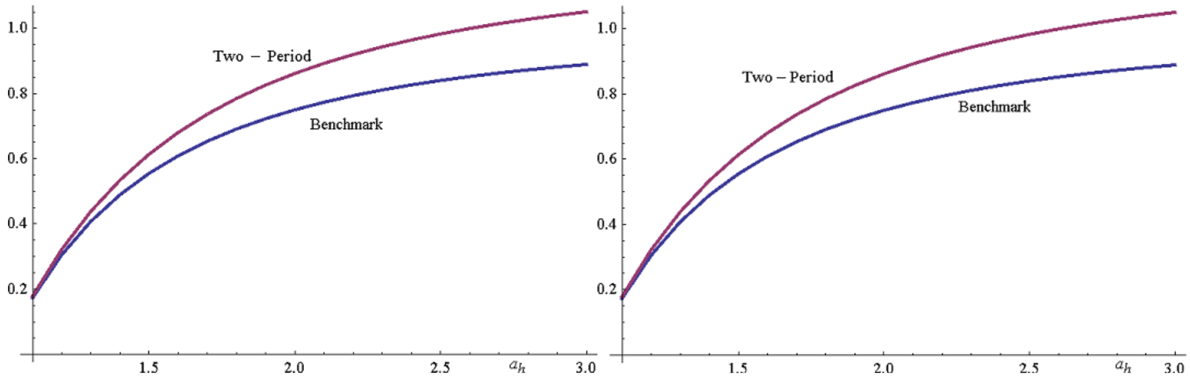

Figure 6: Payoffs of high ability player in terms of ability ratio, cost: $c(e) = e$ and $c(e) = e^2$.

with one half chance in both the two period and in the benchmark game. Therefore, expected payoffs can be written as

$$E[\pi_1 + \pi_2] = 1 - E[c(x_1) + c(x_2)] = \frac{k_h}{2} > \frac{1}{2A} = 1 - E[2c(x)] = E[2\pi_b]$$

This implies that $E[c(x_1) + c(x_2)] < E[2c(x)]$. Also, since $c(\cdot)$ is weakly convex, then

$$E\left[2c\left(\frac{x_1 + x_2}{2}\right)\right] \leq E[c(x_1) + c(x_2)] < E[2c(x)].$$

Because $c(\cdot)$ is strictly increasing, this implies that

$$E[x_1 + x_2] < E[2x]$$

and therefore output in the two period game is lower than the two period benchmark where no information is revealed after the first round. □

From the incentives driven by the continuation values, it is clear that in the first contest, high ability players have an incentive to reduce effort and appear weaker. These players have a lower expected output in the first of two contests than in the single contest benchmark. On the other hand, low ability players have an incentive to appear stronger, which increases their expected output above the level of the benchmark. When the abilities of players are sufficiently different, players' ex-ante expected outputs are lower than in the benchmark as the effect of the high ability players outweighs that of the low ability players. Intuitively, since the output of high ability players is higher for a given level of effort relative to a low ability player, then changes in their effort result in a greater change in output.

The reduction of effort in the second contest stems from increased differentiation of players. With a high probability, one player will enter the second contest in a stronger position than his opponent. The difference in positions will reduce competition and on average, less total output will be produced. The weaker player faces a negative motivation effect, while the stronger player will not compete as aggressively against a weaker opponent.

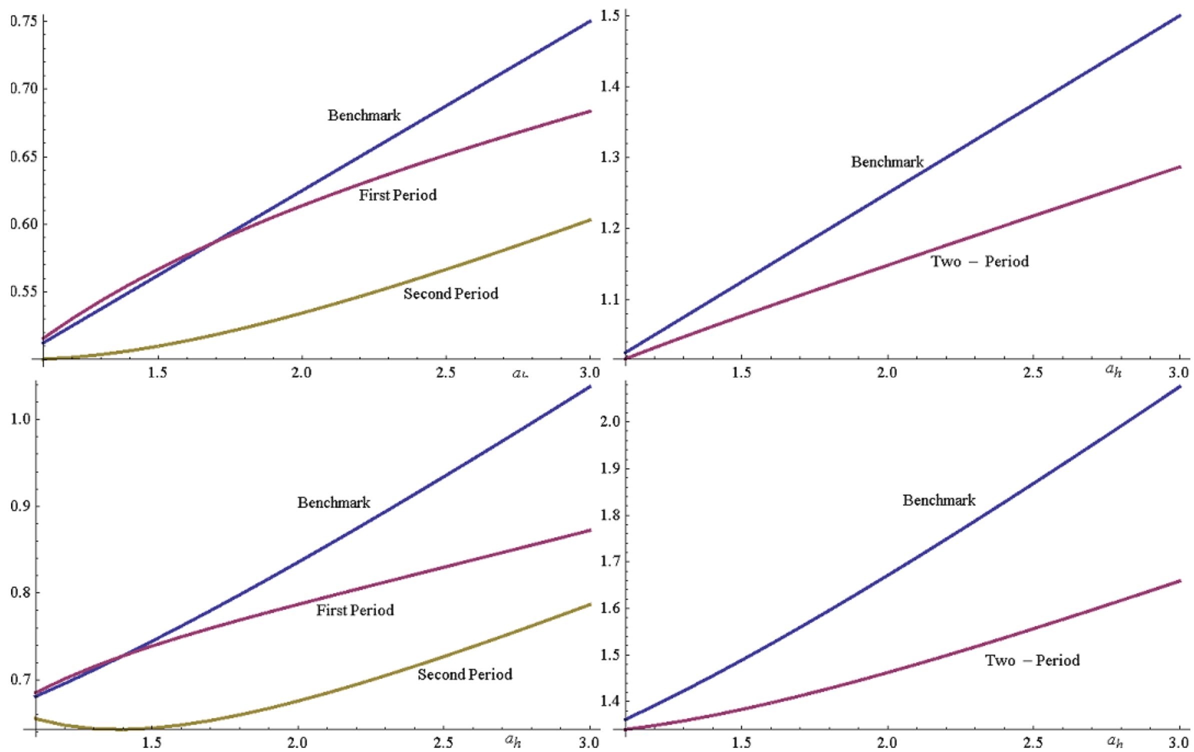

Figure 7: Output in terms of ability ratio, cost: $c(e) = e$ and $c(e) = e^2$.

5.3.1 Application to Performance Evaluations

Ederer (2010) and Aoyagi (2010) both discuss the potential merits of performance evaluations in a single contest between two employees. Performance evaluations can be thought of as dividing a contest into two separate contests, where agents choose a level of effort before and after the evaluation. Aoyagi (2010) shows that when output is a noisy signal of effort and abilities do not effect output, performance evaluations reduce the expected output of the workers if the cost of effort is convex. On the other hand, Ederer (2010) shows that when ability affects output and the contestants do not know their ability, there are two competing effects of performance evaluations: the "evaluation effect" which stems from relative position in the contest and the "motivation effect" which which encourages the contestant who appears more productive. Strategically, the evaluation effect discourages the employee who is further behind while the motivation effect discourages employees who think they are less productive. This motivation effect also provides additional incentive for effort before the performance evaluation is administered as the employee wants to appear more productive to his opponents. It is shown that this additional effort before the performance evaluation may outweigh the loss in output from the decreased competition after the evaluation.

Our results indicate that when employees have private information about their abilities, that the effect in the first period is not one directional. After the midterm evaluation a high ability employee would actually prefer to look weaker, and therefore will produce less effort before the evaluation. This would effectively counteract any additional effort exerted by low ability employees before the midterm evaluation. Additionally, after the evaluation, both differentiation between employees abilities and the discouragement effect stemming from one employee falling behind will combine to reduce expected output. Therefore, in this setting, performance evaluations would not encourage additional effort from employees.

6 Conclusion

While competing in two repeated contests with asymmetric information, contestants have an incentive to give up potential profits in the first contest to prevent revealing their private information. This leads to both bluffing and sandbagging in the first contest and can cause the following inefficiencies as compared to the single contest benchmark. First, a low ability player has a positive probability of winning both contests against a player who has high ability. Second, repeated contests have a lower expected output than the single contest, and additionally, the expected output of the second contest is lower than that of the first. While the results may seem overwhelmingly negative, in settings where the payoffs of competitors are of interest, our results are positive as ex-ante expected payoffs are higher for the contestants. Additionally, we feel that the intuitions developed here apply in more general dynamic settings where private information is valuable, and we leave that for future work.

Appendix

A Equilibrium Construction

Single Contest

Because the equilibrium is monotonic, we know $BR_i(a_h) = x^*$ for $i = s, w$. Additionally, each contestant must be indifferent between all $x \in (x_i^*, x^*)$ when they have high ability. Given high ability these contestants have the same marginal cost of output, and therefore the density of the expected output of their opponents must also be the some for both indifference conditions to hold. Therefore, $f_s^*(x) = f_w^*(x)$ for $x \in (\max\{x_s^*, x_w^*\}, x^*)$ and $F_s^*(x^*) = F_w^*(x^*) = 1$. Since $f_i^*(x) = \mu_i h_i^*(x)$ for all $x \in (x_i^*, x^*)$, then $h_s^*(x) \leq h_w^*(x)$ for all $x \in (\max\{x_s^*, x_w^*\}, x^*)$. Also, $H_i(x_i^*) = 0$, which implies that $x_s^* \leq x_w^*$. Therefore, for the remainder of the construction, there are three intervals to consider: the best response set of the low types of both the stronger and the weaker players, $0 \leq x \leq x_s^*$, the best response set of the low type of the weaker player and the high type of the strong type, $x_s^* \leq x \leq x_w^*$, and best response set of the high types of each player, $x_w^* \leq x \leq x^*$.

Within their best response sets, players must be indifferent between all output levels. For example, player s given that he has ability of a_h, must be indifferent to picking all outputs between x_s^* and x^*. Then, for any for any x and x' in this interval the payoffs for the strong contestant must be the same. This puts a condition on $H_w(x)$, the output distribution of the weak contestant with high ability on the interval $[x_w^*, x^*]$, as the indifference for the strong contestant implies

$$\mu_w H_w^*(x') - c\left(\frac{x}{a_h}\right) = \mu_w H_w^*(x') - c\left(\frac{x'}{a_h}\right).$$

Rearranging and taking the limit as $x \to x'$,

$$\lim_{x\to x'} \frac{H_w^*(x) - H_w^*(x')}{c(\frac{x}{a_h}) - c(\frac{x'}{a_h})} = \frac{\partial H_w^*}{\partial c(\frac{x'}{a_h})} = \frac{1}{\mu_w}.$$

We use this to calculate the output density of the weak contestant on this interval.

$$\lim_{x\to x'} \frac{H_w^*(x) - H_w^*(x')}{x - x'} = \lim_{x\to x'} \frac{H_w^*(x) - H_w^*(x')}{c(\frac{x}{a_h}) - c(\frac{x'}{a_h})} \frac{c(\frac{x}{a_h}) - c(\frac{x'}{a_h})}{a_h(\frac{1}{a_h}(x - x'))}$$

$$h_w^*(x') = \frac{\partial H_w^*}{\partial c(\frac{x'}{a_h})} c'\left(\frac{x'}{a_h}\right)\frac{1}{a_h} = \frac{c'(x'/a_h)}{a_h \mu_w}$$

A similar calculation on each interval for each player allows us to characterize the densities of the output on each of the intervals below.

- $x_w^* \le x \le x^*$: $h_s^*(x) = \frac{c'(x/a_h)}{a_h\mu_s}$, $h_w^*(x) = \frac{c'(x/a_h)}{a_h\mu_w}$, $f_s^*(x) = f_w^*(x) = \frac{c'(x/a_h)}{a_h}$.
- $x_s^* \le x \le x_w^*$: $h_s^*(x) = \frac{c'(x)}{\mu_s}$, $\ell_w^*(x) = \frac{c'(x/a_h)}{a_h(1-\mu_w)}$, $f_s^*(x) = c'(x)$, $f_w^*(x) = \frac{c'(x/a_h)}{a_h}$.
- $0 \le x \le x_s^*$: $\ell_s^*(x) = \frac{c'(x)}{1-\mu_s}$, $\ell_w^*(x) = \frac{c'(x)}{1-\mu_w}$, $f_s^*(x) = f_w^*(x) = c'(x)$.

From the definition of the best response sets and the consistency of player's information sets, the distribution of output for each player must satisfy

$$L_i^*(x_i^*) = 1, \quad H_i^*(x_i^*) = 0, \quad F_i^*(x_i^*) = 1 - \mu_i, \quad F_i^*(x^*) = 1.$$

To find the endpoints we look at the stronger player's distribution of strategies. The stronger contestant does not choose zero effort with positive probability, and therefore $L_s^*(0) = 0$. Using $L_s^*(x_s^*) = 1$ and the definition of $\ell_s^*(x)$ on $[0, x_s^*]$, we calculate x_s^*.

$$\int_0^{x_s^*} \ell_s^*(x)dx = L_s^*(x_s^*) - L_s(0) = \frac{c(x_s^*)}{1-\mu_s} = 1$$

Then $c(x_s^*) = 1 - \mu_s$, so that $x_s^* = c^{-1}(1-\mu_s)$. Similarly, $x_w^* = c^{-1}(1-\mu_w)$. From these endpoints we can calculate x^*.

$$\int_{x_s^*}^{x_w^*} h_s^*(x)dx = \frac{c(x_w^*) - c(x_s^*)}{\mu_s} = \frac{(1-\mu_w) - (1-\mu_s)}{\mu_s} = \frac{\mu_s - \mu_w}{\mu_s}$$

$$\int_{x_w^*}^{x^*} h_s^*(x_s)dx = 1 - \frac{\mu_s - \mu_w}{\mu_s} = \frac{\mu_w}{\mu_s}$$

$$\int_{x_w^*}^{x^*} f_s^*(x_s)dx = c\left(\frac{x^*}{a_h}\right) - c\left(\frac{c^{-1}(1-\mu_w)}{a_h}\right) = \mu_w$$

$$x^* = a_h c^{-1}\left(\mu_w + c\left(\frac{c^{-1}(1-\mu_w)}{a_h}\right)\right)$$

Lastly, we can pin down the probability that the weaker player exerts no effort.

$$\int_{x_s^*}^{x_w^*} \ell_w^*(x)dx = \frac{1}{1-\mu_w}\left[c\left(\frac{c^{-1}(1-\mu_w)}{a_h}\right) - c\left(\frac{c^{-1}(1-\mu_s)}{a_h}\right)\right]$$

$$\int_0^{x_s^*} \ell_w^*(x)dx = \frac{c(c^{-1}(1-\mu_s))}{1-\mu_w} - 0 = \frac{1-\mu_s}{1-\mu_w}$$

$$\begin{aligned}L_w^*(0) &= 1 - \frac{1-\mu_s}{1-\mu_w} - \frac{1}{1-\mu_w}\left[c\left(\frac{c^{-1}(1-\mu_w)}{a_h}\right) - c\left(\frac{c^{-1}(1-\mu_s)}{a_h}\right)\right]\\ &= \frac{\mu_s - \mu_w - \left[c\left(\frac{c^{-1}(1-\mu_w)}{a_h}\right) - c\left(\frac{c^{-1}(1-\mu_s)}{a_h}\right)\right]}{1-\mu_w}\end{aligned}$$

The endpoints of the best response sets for each type of each player, and characterizations of the density functions within these best response sets characterize the unique equilibrium.

Repeated Contests

We solve for the equilibrium of successive contest given a parameterization of the cost function, $c(x) = kx^\alpha$. The original assumptions on the cost function imply that $\alpha \geq 1$ and $k > 0$.

For the range of $0 \leq x < x_{h,*}$ we have $F_1^*(x) = kx^\alpha$, and for the range $x_\ell^* < x \leq x_h^*$, we have $F_1^*(x) + v_h = k\frac{x^\alpha}{a_h^\alpha} + k_h$.

For the range $x_{h,*} \leq x \leq x_\ell^*$, the solution to (†) is

$$F_1^*(x) = Be^{c(x/a_h)-c(x)} + \frac{\frac{\partial}{\partial x}c(x)}{\frac{\partial}{\partial x}c(x) - \frac{\partial}{\partial x}c(x/a_h)}, \text{ with } F_1(x_{h,*}) = kx_{h,*}^\alpha.$$

Solving for, B, the ex-ante distribution function of each player on $[x_{h,*}, x_\ell^*]$ is

$$F_1^*(x) = \frac{a_h^\alpha}{a_h^\alpha - 1} - \left(\frac{a_h^\alpha}{a_h^\alpha - 1} - kx_{h,*}^\alpha\right)e^{\frac{k(1-a_h^\alpha)}{a_h{}^\alpha}(x^\alpha - x_{h,*}^\alpha)}.$$

The belief function must satisfy

$$d(\mu(x))\mu'(x) = c'(x) - \frac{1}{a_h}c'(x/a_h) \text{ for } x \in [x_{h,*}, x_\ell^*], \text{ with } \mu(x_{h,*}) = 0.$$

Therefore, on this interval, the belief function is $\mu(x) = k(x^\alpha - x_{h,*}^\alpha)$, and the distribution function can be written as

$$F_1^*(x) = \frac{a_h^\alpha}{a_h^\alpha - 1} - \left(\frac{a_h^\alpha}{a_h^\alpha - 1} - kx_{h,*}^\alpha\right)e^{-\frac{(a_h^\alpha - 1)}{a_h^\alpha}\mu(x)}.$$

Given $F_1^*(x)$ and $\mu(x)$ we can also calculate the output distribution of the both the high and low ability players on $[x_{h,*}, x_\ell^*]$, using $2F_1^*(x) = L_1^*(x) + H_1^*(x)$ and $\mu(x) = \frac{h_1^*(x)}{2f_1^*(x)}$.

$$
\begin{aligned}
H_1^*(x) &= H_1^*(x_{h,*}) + 2\int_{x_{h,*}}^{x} \mu(t) f_1^*(t)dt \\
&= 2\mu(t)F_1^*(t)|_{x_{h,*}}^{x} - 2\int_{x_{h,*}}^{x} \mu'(t)F_1^*(t)dt \\
&= 2\left(\frac{a_h^\alpha}{a_h^\alpha - 1} - \left(\frac{a_h^\alpha}{a_h^\alpha - 1} + \mu(x)\right) e^{\frac{1-a_h^\alpha}{a_h^\alpha}\mu(x)}\right)\left(\frac{a_h^\alpha}{a_h^\alpha - 1} - kx_{h,x}^\alpha\right) \\
L_1^*(x) &= 2F_1^*(x) - H_1^*(x) \\
&= \frac{2a_h^\alpha}{a_h^\alpha - 1} + 2\left(\left(\mu(x) + \frac{a_h^\alpha}{a_h^\alpha - 1} - 1\right) e^{\frac{1-a_h^\alpha}{a_h^\alpha}\mu(x)} - \frac{a_h^\alpha}{a_h^\alpha - 1}\right)\left(\frac{a_h^\alpha}{a_h^\alpha - 1} - kx_{h,x}^\alpha\right)
\end{aligned}
$$

Let $A = \frac{a_h^\alpha}{a_h^\alpha - 1}$. Ex- ante and ex-interim strategies in the first contest are

$$
F_1^*(x) = \begin{cases} kx^\alpha, & 0 \le x \le x_{h,*} \\ A - (A - kx_{h,*}^\alpha)e^{-\mu(x)/A}, & x_{h,*} \le x \le x_\ell^* \\ k(x/a_h)^\alpha + k_h - v_h, & x_\ell^* \le x \le x_h^* \end{cases}
$$

$$
L_1^*(x) = \begin{cases} 2kx^\alpha, & 0 \le x \le x_{h,*} \\ 2A + 2((\mu(x) + A - 1)e^{-\mu(x)/A} - A)(A - kx_{h,*}^\alpha), & x_{h,*} \le x \le x_\ell^* \\ 1, & x_\ell^* \le x \le x_h^* \end{cases}
$$

$$
H_1^*(x) = \begin{cases} 0, & 0 \le x \le x_{h,*} \\ 2(A - (A + \mu(x))e^{-\mu(x)/A}(A - kx_{h,*}^\alpha), & x_{h,*} \le x \le x_\ell^* \\ 2k(x/a_h)^\alpha + 2(k_h - v_h) - 1, & x_\ell^* \le x \le x_h^* \end{cases}
$$

We use the following conditions to find the unknowns, $x_{h,*}$, x_ℓ^*, x_h^*, v_h and k_h:

1. Continuity of the belief function implies that $\mu(x_\ell^*) = 1$ and $k(x_\ell^{*\alpha} - x_{h,*}^\alpha) = 1$.
2. Since $x_\ell^* = \sup\{BR(a_\ell)\}$, then $L_1^*(x_\ell^*) = 1$.

$$
L_1^*(x_\ell^*) = 2A + 2\left((\mu(x_\ell^*) + A - 1)e^{-\mu(x_\ell^*)/A} - A\right)(A - kx_{h,*}^\alpha) = 1
$$

$$
\Rightarrow kx_{h,*}^\alpha = A - \frac{2A-1}{2A(1-e^{-1/A})}, \quad kx_\ell^{*\alpha} = 1 + A - \frac{2A-1}{2A(1-e^{-1/A})}
$$

3. Continuity of $F_1^*(x)$ at x_ℓ^* gives

$$
A - (A - kx_{h,*}^\alpha)e^{-1/A} = \frac{kx_\ell^{*\alpha}}{a_h^\alpha} + k_h - v_h.
$$

Substituting from above we get the two period payoff of the high type player

$$
k_h = v_h + \frac{1}{A} + \frac{(2A-1)(1-a_h^\alpha e^{-1/A})}{2Aa_h^\alpha(1-e^{-1/A})}.
$$

4. v_h is the expected payoff in the second period of a player with high ability who reveals that he is high type in the first period.

$$v_h = E[v_i(1, \mu(x^{-i}), a_h)] = \frac{1}{A} E[1 - \mu(x^{-i})] = \frac{1}{2A}$$

$$\Rightarrow k_h = \frac{3}{2A} + \frac{(2A-1)(1 - a_h^\alpha e^{-1/A})}{2Aa_h^\alpha (1 - e^{-1/A})}.$$

5. The last condition is using that $x_h^* = \sup BR(a_h)$, so that $F_1^*(x_h^*) = 1$.

$$F_1^*(x_h^*) = \frac{k x_h^{*\,\alpha}}{a_h^\alpha} + k_h - v_h = 1$$

$$\Rightarrow k x_h^{*\,\alpha} = 1 - \frac{(2A-1)(1 - a_h^\alpha e^{-1/a})}{2A(1 - e^{-1/A})}$$

B Proofs

Lemma 3.1 *In any equilibrium, players' distributions of output, $H_s(x), L_s(x), H_w(x)$, and $L_w(x)$, are continuous on $(0, x^*)$, where $x^* \equiv \sup\{BR_s(a_\ell) \cup BR_s(a_h)\} = \sup\{BR_w(a_\ell) \cup BR_w(a_h)\}$ and $\inf\{BR_s(a_\ell) \cup BR_s(a_h)\} = \inf\{BR_w(a_\ell) \cup BR_w(a_h)\} = 0$.*

Proof. The proof follows in four steps:

1. There is no x at which both players both have an atom.
   If both players played some x with positive probabilities given by p_1 and p_2. Then either player can increase output slightly above x, to $x + \varepsilon$. This would increase the payoff of that player since the cost of effort is continuous and we can pick ε such that $c(x+\varepsilon) - c(x) < p_2$. However, this implies that x is not a best response of that player, a contradiction.

2. If a player has an atom, then it is at zero.
   Assume that player i has an atom at $x > 0$ where x is played is probability $p > 0$. Then by the continuity of the cost function in output, there is a $\delta > 0$ such that $\hat{x} \in (x - \delta/2, x)$, $\hat{x} \notin BR_{-i}(a^{-i})$. This implies, that player i would do better by playing $x - \delta/4$, and therefore $x \notin BR_i(a^i)$. This is a contradiction. Therefore the output distribution functions of each type of each player is continuous on $(0, \infty)$. This implies that $F_s(x)$, $F_w(x)$

3. If $x > 0$ is not a best response for any ability of one of the contestants, then for all $x' > x$, x' is not a best response for either type of either player.
   Step (2) implies that payoffs are continuous, since both the cost function and the probability of winning are continuous in x. Now, since $x \notin \{BR_i(a_\ell) \cup BR_i(a_h)\}$, for some $i = s, w$, $\exists \tilde{x}(a_h), \tilde{x}(a_\ell)$ for which $\pi^i(\tilde{x}(a_h), a_h) > \pi^i(x, a_h) + \varepsilon$ and $\pi^i(\tilde{x}(a_\ell), a_\ell) > \pi^i(x, a_\ell) + \varepsilon$. Then, there is a $\delta > 0$ for which $\pi^i(\tilde{x}(a_h), a_h) > \pi^i(\hat{x}, a_h)$ and $\pi^i(\tilde{x}(a_\ell), a_\ell) > \pi^i(\hat{x}, a_\ell)$, $\forall \hat{x} \in (x, x + \delta)$. Therefore every $\hat{x}$ in this neighborhood cannot be a best response of either type of player i. Additionally,

no $\hat{x}$ in this neighborhood can be a best response for any type of player $-i$, as they could improve utility by lowering output to x. Therefore there is a interval with positive measure for which there is no best responses for either player for either type. Let X^* be the set of all outputs that are greater than x and are a best response for some player of either type. Let $x_* = \inf\{X^*\}$. Then, necessarily x_* has a gap $(x_* - \delta', x_*]$, $\delta' > 0$ for which there are no best responses. However, since there is an $x \in X^*$ such that $x - x_* < \delta$, x cannot be a best response. Therefore, x_* does not exist and X^* is empty. This implies that $\sup\{BR_s(a_\ell) \cup BR_s(a_h)\} = \sup\{BR_w(a_\ell) \cup BR_w(a_h)\}$. We call this output level x^*.

4. Each player has a type who has best response that is arbitrarily close to 0.
If this were not true, then there is a player and an $x > 0$ such that all $\hat{x} \leq x$ are not a best response for any type of that player. Then from step (3), that player has no best responses. This cannot be true in equilibrium.

□

Proposition 4.1

(Countervailing Incentives) $\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_h)] < 0$ *and* $\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_\ell)] > 0$ *for all* $\mu_i \in (0, 1)$.

Proof. In the second contest, for a given pair of beliefs, players will expect the following payoffs:

$$v_i(\mu_i, \mu_{-i}, a_h) = 1 - \min\{\mu_i, \mu_{-i}\} - c\left(\frac{c^{-1}(1 - \min\{\mu_i, \mu_{-i}\})}{a_h}\right)$$

$$v_i(\mu_i, \mu_{-i}, a_\ell) = \begin{cases} \mu_i - \mu_{-i} - \left[c\left(\frac{c^{-1}(1-\mu_{-i})}{a_h}\right) - c\left(\frac{c^{-1}(1-\mu_i)}{a_h}\right)\right] & \text{if } \mu_i \geq \mu_{-1} \\ 0 & \text{otherwise} \end{cases}$$

For a high ability contestant whose opponent has belief μ_i, the expected payoff in the second contest is given by

$$E[v_i(\mu_i, \mu_{-i}, a_h)] = \int_0^1 \left(1 - \min\{\mu_i, \mu_{-i}\} - c\left(\frac{c^{-1}(1 - \min\{\mu_i, \mu_{-i}\})}{a_h}\right)\right) dF_{\mu_{-i}}(\mu_{-i})$$

As the opponent believes the contestant is stronger, the change in expected payoff is

$$\frac{\partial}{\partial \mu_i} E_{\mu_{-i}}[v_i(\mu_i, \mu_{-i}, a_h)] = \left(1 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(1 - \mu_i)}{a_h}\right)\right)(F_{\mu_{-i}}(\mu_i) - 1)$$

For a low ability contestant, the expected payoff given μ_i is:

$$E[v_i(\mu_i, \mu_{-i}, a_\ell)] = \int_0^{\mu_i} \left(\mu_i - \mu_{-i} + c\left(\frac{c^{-1}(1 - \mu_i)}{a_h}\right) - c\left(\frac{c^{-1}(1 - \mu_{-i})}{a_h}\right)\right) dF_{\mu_{-i}}(\mu_{-i})$$

The effect of his opponent's beliefs is

$$\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_\ell)] = \left(1 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(1-\mu_i)}{a_h}\right)\right) F_{\mu_{-i}}(\mu_i)$$

Given the assumptions on the cost of effort, $c'(e) > 0$ and $c''(e) \geq 0$,

$$\frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(1-\mu_i)}{a_h}\right) = -\frac{1}{a_h} c'\left(\frac{c^{-1}(1-\mu_i)}{a_h}\right) \frac{1}{c'(c^{-1}(1-\mu_i))} \in \left[-\frac{1}{a_h}, 0\right).$$

If we define,

$$d(\mu_i) \equiv \left[1 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(1-\mu_i)}{a_h}\right)\right] \text{ where } d(\mu_i) \in \left[\frac{a_h - 1}{a_h}, 1\right) \text{ for all } \mu_i,$$

then it is clear that

$$\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_h)] = d(\mu_i)(F_{\mu_{-i}}(\mu_i) - 1) < 0$$

$$\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_\ell)] = d(\mu_i) F_{\mu_{-i}}(\mu_i) > 0.$$

□

Corollary 4.2 *In every SPBE, $\mu(x)$ is weakly increasing in x for all $x \in X_1 = X_1^h \cap X_1^\ell$.*

Proof. Assume otherwise, namely that for a given x and y which are best responses for some ability level we have that $x < y$ and $\mu(x) > \mu(y)$. This implies that $0 \leq \mu(y) < \mu(x) \leq 1$. Then by Bayes' Rule, $h_1(x) > 0$ and $h_1(y) < f_1(y)$, which implies that $\ell_1(y) > 0$. For the strategies to be optimal it must be that $x \in BR(a_h)$ and $y \in BR(a_\ell)$. Then we know that

$$\begin{aligned}
\Pr(\text{win}|y) - c(y) + &E[v_i(\mu(y), \mu(x^{-i}), a_\ell)] \\
&\geq \Pr(\text{win}|x) - c(x) + E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] \\
\Pr(\text{win}|y) - c(y/a_h) + &E[v_i(\mu(y), \mu(x^{-i}), a_h)] \\
&\leq \Pr(\text{win}|x) - c(x/a_h) + E[v_i(\mu_i(x), \mu(x^{-i}), a_h)]
\end{aligned}$$

This implies that

$$\begin{aligned}
\Pr(\text{win}|y) - \Pr(\text{win}|x) + &E[v_i(\mu(y), \mu(x^{-i}), a_\ell)] - E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] \\
&\geq c(y) - c(x) \\
\Pr(\text{win}|y) - \Pr(\text{win}|x) + &E[v_i(\mu(y), \mu(x^{-i}), a_h)] - E[v(\mu(x), \mu(x^{-i}), a_h)] \\
&\leq c(y/a_h) - c(x/a_h)
\end{aligned}$$

The expected payoff in the second contest increases for a low ability contestant as μ increases and for a high ability contestant decreases as μ increases. Then $\mu(x) > \mu(y)$ implies

$$E[v_i(\mu(y), \mu(x^{-i}), a_\ell)] - E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] < 0,$$
$$\text{and } E[v_i(\mu(y), \mu(x^{-i}), a_h)] - E[v(\mu(x), \mu(x^{-i}), a_h)] > 0.$$

Combine with previous inequalities:

$$c(y) - c(x) < \Pr(\text{win}|y) - \Pr(\text{win}|x) < c(y/a_h) - c(x/a_h)$$

However, since $c''(x) \geq 0$ and $c'(x) > 0$, we must have that

$$\begin{aligned} c(y/a_h) - c(x/a_h) &\leq c(y) - c(y - (y/a_h - x/a_h)) \\ &= c(y) - c\left(\frac{x + (a_h - 1)y}{a_h}\right) < c(y) - c(x). \end{aligned}$$

This is a contradiction. □

Lemma 4.4 *There is no output that is played with positive probability and* $\Pr(win|x) = F(x)$ *is continuous.*

Proof. In a symmetric equilibrium, if an output is played with positive probability by one type of player, then it must be played with positive probability by this type of both players. Let $\hat{x} \in \{X_1^\ell \cup X_1^h\}$ be played with probability $p > 0$. Then

$$\Pr(\text{win}|\hat{x}) + \frac{p}{2} \leq \Pr(\text{win}|x) \text{ for all } x > \hat{x}.$$

Since for some a, $\hat{x} \in BR(a)$, then $\pi(\hat{x}|a) \geq \pi(x|a)$ for all x. This implies that

$$\begin{aligned} &\Pr(\text{win}|\hat{x}) - c(\hat{x}/a) + E[v_i(\mu(\hat{x}), \mu(x^{-i}), a^i)] \\ &\qquad \geq \Pr(\text{win}|x) - c(x/a^i) + E[v_i(\mu(x), \mu(x^{-i}), a^i)] \end{aligned}$$

Combing the above inequalities we have

$$\begin{aligned} \frac{p}{2} &\leq \Pr(\text{win}|x) - \Pr(\text{win}|\hat{x}) \\ &\leq E[v_i(\mu(\hat{x}), \mu(x^{-i}), a^i)] - E[v_i(\mu(x), \mu(x^{-i}), a^i)] + c(x/a^i) - c(\hat{x}/a^i) \end{aligned}$$

By continuity of the cost function, $\exists \varepsilon > 0$ such that for all $x \in (\hat{x}, \hat{x} + \varepsilon)$, we have $c\left(\frac{\hat{x}+\varepsilon}{a^i}\right) - c\left(\frac{\hat{x}}{a^i}\right) < \frac{p}{2}$. Then for each x in this range we know

$$E[v_i(\mu(\hat{x}), \mu(x^{-i}), a^i)] - E[v_i(\mu(x), \mu(x^{-i}), a^i)] > 0.$$

If $a^i = a_\ell$, then $\mu(\hat{x}) > \mu(x)$ and therefore $\hat{x} \in \{X_1^\ell \cap X_1^h\}$. Similarly, if $a^i = a_h$, then $\mu(\hat{x}) < \mu(x)$ and $\hat{x} \in \{X_1^\ell \cap X_1^h\}$. In either case, $\hat{x} \in \{BR(1) \cap BR(a_h)\}$. However, the inequality cannot hold for both $a^i = a_\ell$ and $a^i = a_h$ at the same time, so we have a contradiction. □

We now can use the fact that $F_1(x)$ is continuous in x and we have that $\Pr(\text{win}|x) = \Pr(x < x^{-i}) = \Pr(x \leq x^{-i}) = F_1(x)$. Combined with Lemma 4.2, we have $Pr(\mu(x) < \mu(y)) \leq \Pr(\text{win}|y) = F(y) \leq \Pr(\mu(x) \leq \mu(y))$.

Lemma 4.5 *$BR(a_\ell)$ and $BR(a_h)$ are intervals where $0 = x_{\ell,*} \leq x_{h,*} < x^*_\ell \leq x^*_h$ and we define $x_{\ell,*} = \inf\{BR(a_\ell)\}, x^*_\ell = \sup\{BR(a_\ell)\}, x_{h,*} = \inf\{BR(a_h)\}$ and $x^*_h = \sup\{BR(a_h)\}$.*

Proof. The proof follows in four steps.

1. We first show that $x_{\ell,*} = 0$. We do this by first showing that $x_{\ell,*} \leq x_{h,*}$, and then showing that $x_{\ell,*}$ cannot be larger than zero.

   Let $x_{h,*} < x_{\ell,*}$. Since $x_{h,*} = \inf\{X_1^h\}$, $\forall \varepsilon > 0$, $\exists x_\varepsilon$ such that $x_{h,*} \leq x_\varepsilon < x_{h,*} + \varepsilon$ and $x_\varepsilon \in X_1^h$. In particular, this holds for $\varepsilon^* = x_{\ell,*} - x_{h,*}$. Then $x_{\varepsilon^*} \in \{X_1^h \setminus X_1^\ell\}$ and $\mu(x_{\varepsilon^*}) = 1$. However, from Lemma 3.1 we would have $\mu(x) = 1$ for all $x \in X_1^\ell$, which cannot hold. Therefore $x_{h,*} \geq x_{\ell,*}$.

   If $0 < x_{\ell,*} < x_{h,*}$, then let $x_{h,*} - x_{\ell,*} = \delta_1$. Since F_1 is continuous from Lemma 3.2, then $\exists \delta_2$ with $0 < \delta_2 < \delta_1$ such that $\forall x \in (x_{\ell,*}, x_{\ell,*} + \delta_2)$ we have $|F_1(x) - F_1(0)| = |F_1(x) - F_1(x_{\ell,*})| < c(x_{\ell,*}) < c(x_{\delta_2})$. Let $x_{\delta_2} \in X_1^\ell \cap (x_{\ell,*}, x_{\ell,*} + \delta_2)$. Then $\mu(x_{\delta_2}) = 0$ and

$$\begin{aligned}E[\pi^i(0)|a_\ell] &= F_1(0) + E[v_i(\mu(0), \mu(x^{-i}), a_\ell)] \\ &> F_1(x_{\delta_2}) + E[v_i(\mu(x_{\delta_2}), \mu(x^{-i}), a_\ell)] - c(x_{\delta_2}) \\ &= E[\pi^i(x_{\delta_2})|a_\ell]\end{aligned}$$

   Then $x_{\delta_2} \notin BR(a_\ell)$, a contradiction.

   If $0 < x_{\ell,*} = x_{h,*}$, then $\exists x_\ell, x_h$ such that $x_\ell \leq x_h$, $x_\ell \in X_1^\ell$, $x_h \in X_1^h$, and $F_1(x_\ell) - F_1(x_{\ell,*}) = F_1(x_\ell) < c(x_{\ell,*}) < c(x_\ell)$ and $F_1(x_h) - F_1(x_{h,*}) = F_1(x_h) < c(x_{h,*}/a_h) < c(x_h/a_h)$, by the continuity of F_1.
   $x_\ell \in X_1^\ell$ implies that

$$\begin{aligned}E[\pi^i(x_\ell)|a_\ell] &= F_1(x_\ell) - c(x_\ell) + E[v_i(\mu(x_\ell), \mu(x^{-i}), a_\ell)] \\ &\geq F_1(0) - c(0) + E[v_i(\mu(0), \mu(x^{-i}), a_\ell)] = E[\pi^i(0)|a_\ell]\end{aligned}$$

   This can hold only if $E[v_i(\mu(x_\ell), \mu(x^{-i}), a_\ell)] > E[v_i(\mu(0), \mu(x^{-i}), a_\ell)]$, which implies that $\mu(x_\ell) > \mu(0)$.
   $x_h \in X_1^h$ implies that

$$\begin{aligned}E[\pi^i(x_2)|a_h] &= F_1(x_h) - c(x_h/a_h) + E[v_i(\mu(x_h), \mu(x^{-i}), a_h)] \\ &\geq F_1(0) - c(0) + E[v_i(\mu(0), \mu(x^{-i}), a_h)] = E[\pi^i(0)|a_h]\end{aligned}$$

   This can hold only if $E[v_i(\mu(x_h), \mu(x^{-i}), a_h)] > E[v_i(\mu(0), \mu(x^{-i}), a_h)]$, which implies that $\mu(x_h) < \mu(0)$.
   Combining these two inequalities leads to $\mu(x_h) < \mu(x_\ell)$. This contradicts Lemma 4.2.
   Therefore we must have $0 = x_{\ell,*} \leq x_{h,*}$.

2. We next show that $x_{h,*} \le x^*_\ell$.

   If $x^*_\ell > x_{h,*}$, then $\forall x \in (x^*_\ell, x_{h,*})$, $x \notin \{X_1^\ell \cap X_1^h\}$. Let $x' = \frac{x^*_\ell + x_{h,*}}{2}$ and $\varepsilon = c(x_{h,*}/a_h) - c(x'/a_h)$. There is a $\delta > 0$ such that $\forall x \in (x_{h,*}, x_{h,*} + \delta)$, $F(x) - F(x_{h,*}) < \varepsilon$. Pick an x_δ such that $x_\delta \in (x_{h,*}, x_{h,*} + \delta)$ and $x_\delta \in X_1^h$. Then $F_1(x_\delta) - F_1(x_{h,*}) = F_1(x_\delta) - F_1(x') < \varepsilon$, $c(x_\delta/a_h) - c(x'/a_h) > \varepsilon$, and $E[v_i(\mu(x_\delta), \mu(x^{-i}), a_h)] \le E[v_i(\mu(x'), \mu(x^{-i}), a_h)]$. Therefore

   $$\begin{aligned} E[\pi^i(x')|a_h] &= F_1(x') + E[v_i(\mu(x'), \mu(x^{-i}), a_h)] - c\left(\frac{x'}{a_h}\right) \\ &> F(x_\delta) + E[v_i(\mu(x_\delta), \mu(x^{-i}), a_h)] - c\left(\frac{x_\delta}{a_h}\right) = E[\pi^i(x_\delta)|a_h], \end{aligned}$$

   a contradiction. So we can conclude that $x^*_\ell \le x_{h,*}$.

   Also note that we must have $x^*_\ell \le x^*_h$. If we assume otherwise, then we can find $x \in \{X_1^\ell \setminus X_1^h\}$ where $x > x^*_h$ and $\mu(x) = 0$. Lemma 4.2 rules out this possibility.

   We have shown so far that $0 = x_{\ell,*} \le x_{h,*} \le x^*_\ell \le x^*_h$.

3. We next will show that for all $x \in (x_{\ell,*}, x_{h,*})$, $x \in BR(a_\ell)$ and for all $x \in (x^*_\ell, x^*_h)$, $x \in BR(a_h)$.

   If $x_{\ell,*} < x_{h,*}$, then let $X_c^\ell = \{x | x \in \{(x_{\ell,*}, x_{h,*}) \setminus BR(a_\ell)\}\}$. If $x \in X_c^\ell$, then $\exists \varepsilon > 0$ such that $E[\pi^i(x)|a_\ell] < E[\pi^i(x')|a_\ell] - \varepsilon$ for all $x' \in \{(x_{\ell,*}, x_{h,*}) \cap X_1^\ell\}$. This implies that:

   $$F_1(x) + E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] - c(x) < F_1(x') + E[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] - c(x') - \varepsilon,$$

   where $E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] \ge E[v_i(\mu(x'), \mu(x^{-i}), a_\ell)]$ as $\mu(x') = 0$. Therefore $F_1(x) - c(x) < F_1(x') - c(x') - \varepsilon$, and for all $x' > x$ with $x' \in \{(x_{\ell,*}, x_{h,*}) \cap X_1^\ell\}$, $F_1(x') - F_1(x) > c(x') - c(x) - \varepsilon$.

   Since F_1 and c are continuous, then there is a $\delta(\varepsilon) > 0$ such that for all $x' \in X_1^\ell$, $|x' - x| \ge \delta(\varepsilon)$. This implies that x is contained in an interval which is a subset of X_c^ℓ. Let a and b be the infimum and supremem of this interval respectively.

   – If $b < x_{h,*}$, then $\exists x' < x_{h,x}, x' \in X_1^\ell$ where $|x' - b| < \delta, \forall \delta > 0$. Then, by the continuity of F, $\exists x' \in X_1^\ell$ and $F(x') - F(b) < c(b) - c(\frac{a+b}{2})$. Then we know that

   $$F_1(x') - F_1\left(\frac{a+b}{2}\right) < c(b) - c\left(\frac{a+b}{2}\right) \text{ and}$$
   $$E[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] \le E\left[v_i\left(\mu\left(\frac{a+b}{2}\right), \mu(x^{-i}), a_\ell\right)\right].$$

   This implies that $E[\pi^i(x')|a_\ell] < E[\pi^i(\frac{a+b}{2})|a_\ell]$ which contradicts $x' \in BR(a_\ell)$.

   – If $b = x_{h,*}$, then $\forall \delta > 0$, $\exists x' \in X_1^h$, s.t. $|x' - b| < \delta$. We again can take $x' \in X_1^h$ such that $F_1(x') - F_1(b) < c(\frac{b}{a_h}) - c(\frac{a+b}{2a_h})$.

- $*$ If $x' \notin X_1^\ell$ then $\mu(x') = 1$, but since $E[v_i(\mu(x'), \mu(x^{-i}), a_h)] \leq E\left[v_i\left(\mu\left(\frac{a+b}{2}\right), \mu(x^{-i}), a_h\right)\right]$, then this contradicts $x' \in BR(a_h)$.
- $*$ If $x' \in X_1^\ell$, then $\mu(x') \in (0,1)$. If $\mu(x') \leq \mu(\frac{a+b}{2})$, then this contradicts $x' \in BR(a_\ell)$, but if $\mu(x') \geq \mu(\frac{a+b}{2})$, this contradicts $x' \in BR(a_h)$.

Therefore X_c^ℓ must be empty.

If $x_\ell^* < x_h^*$, then let $X_c^h = \{x | x \in \{(x_\ell^*, x_h^*) \setminus BR(a_h)\}$. If $x \in X_c^h$, then $\exists \varepsilon > 0$ such that $E[\pi(x)|a_h] < E[\pi(x')|a_h] - \varepsilon$ for all $x' \in X_1^h$. This implies that

$$F_1(x) - c\left(\frac{x}{a_h}\right) + E[v_i(\mu(x), \mu(x^{-i}), a_h)] < F_1(x') - c\left(\frac{x'}{a_h}\right) + E[v_i(\mu(x'), \mu(x^{-i}), a_h)] - \varepsilon,$$

where $E[v_i(\mu(x), \mu(x^{-i}), a_h)] \geq E[v_i(\mu(x'), \mu(x^{-i}), a_h)]$ as $\mu(x') = 1$. Therefore $F_1(x) - c(\frac{x}{a_h}) < F_1(x') - c(\frac{x'}{a_h}) - \varepsilon$. Since F_1 and c are continuous, then this holds only if $|x' - x| \geq \delta(\varepsilon) > 0$, $\forall x' \in BR(a_h)$. We take a and b to be the infimum and supremum respectively of the interval of X_c^h containing x. Note that $b < x_h^*$, by the definition of x_h^*.

Now, there is an $x' \in X_1^h$ where $|x' - b| < \delta$ for all $\delta > 0$. Therefore there is an $x' \in BR(a_h)$ such that $F_1(x') - F_1(b) < c(\frac{b}{a_h}) - c(\frac{b+a}{2a_h})$. Note that this implies that $F_1(x') - F_1(\frac{b+a}{2}) < c(\frac{x'}{a_h}) - c(\frac{b+a}{2a_h})$. However, this implies that

$$\begin{aligned} E\left[\pi^i\left(\frac{b+a}{2}\right)|a_h\right] &= F_1\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2a_h}\right) + E\left[v_i\left(\mu\left(\frac{b+a}{2}\right), \mu(x^{-i}), a_h\right)\right] \\ &> F_1(x') - c\left(\frac{x'}{a_h}\right) + E[v_i(\mu(x'), \mu(x^{-i}), a_h)] = E[\pi^i(x')|a_h]. \end{aligned}$$

This contradicts $x' \in BR(a_h)$, and therefore X_c^h must be empty.

4. Lastly, we show that $x_{h,*} < x_\ell^*$, and for all $x \in (x_{h,*}, x_\ell^*)$, $x \in \{BR(a_\ell) \cap BR(a_h)\}$. If $x_\ell^* = x_{h,*}$, then $\forall \delta > 0$, there is $x_\ell \in BR(a_\ell)$ and $x_h \in BR(a_h)$ where $|x_h - x_\ell| < \delta$. Therefore, by the continuity of F_1 and c, there is x_h and x_ℓ for which

$$\begin{aligned} &F_1(x_h) - c\left(\frac{x_h}{a_h}\right) - \left(F_1(x_\ell) - c\left(\frac{x_\ell}{a_h}\right)\right) \\ &< E[v_i(\mu(x_\ell), \mu(x^{-i}), a_h)] - E[v_i(\mu(x_h), \mu(x^{-i}), a_h)] \\ &= E[v_i(0, \mu(x^{-i}), a_h)] - E[v_i(1, \mu(x^{-i}), a_h)], \end{aligned}$$

since $E[v_i(0, \mu(x^{-i}), a_h)] - E[v_i(1, \mu(x^{-i}), a_h)] > 0$. This implies that

$$\begin{aligned} E[\pi^i(x_\ell)|a_h] &= F_1(x_\ell) - c\left(\frac{x_\ell}{a_h}\right) + E[v_i(\mu(x_\ell), \mu(x^{-i}), a_h)] \\ &> F_1(x_h) - c\left(\frac{x_h}{a_h}\right) + E[v_i(\mu(x_h), \mu(x^{-i}), a_h)] = E[\pi^i(x_h)|a_h], \end{aligned}$$

which cannot be true as $x_h \in BR(a_h)$.

Now define $X_c = \{x | x \in (x_{h,*}, x_\ell^*) \setminus (BR(a_\ell) \cup BR(a_h))\}$. From Lemma 3.1, we know that for all $x' \in \{(x_{h,*}, x_\ell^*) \cap (X_1^\ell \cup X_1^h)\}$, $\mu(x') \in (0,1)$ and therefore $x' \in \{X_1^\ell \cap X_1^h\}$. If $\mu(x') = 1$, we must have $x_\ell^* \leq x'$, a contradiction. Similarly if $\mu(x') = 0$, then we must have $x_{h,*} \geq x'$ which is also a contradiction.

Let $x \in X_c$ be given. Then for all $x', x'' \in \{(x_{h,*}, x_\ell^*) \cap (X_1^\ell \cap X_1^h)\}$ such that $x' < x < x''$ we must have $\mu(x') \leq \mu(x'')$. Let $\mu^* \in [\sup\{\mu(x')\}, \inf\{\mu(x'')\}]$. These are well-defined as there is at least one such x' and x''.

If $\mu(x) \geq \mu^*$ then $E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] \geq E[v_i(\mu(x'), \mu(x^{-i}), a_\ell)]$ for all x' as defined above. Therefore

$$F_1(x') - c(x') + E[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] - \varepsilon_1 > F_1(x) - c(x) + E[v_i(\mu(x), \mu(x^{-i}), a_\ell)]$$
$$\Rightarrow F_1(x') - c(x') - \varepsilon_1 > F_1(x) - c(x)$$

Then, by continuity of F_1 and c, $\exists \delta_1 > 0$ such that $\forall x'$, $|x' - x| > \delta_1$. Then $[x - \delta_1, x] \subset X_c$.

If $\mu(x) < \mu^*$, then $E[v_i(\mu(x), \mu(x^{-i}), a_h)] \geq E[v_i(\mu(x''), \mu(x^{-i}), a_h)]$ for all x'' as defined above. Therefore

$$\begin{aligned} F_1(x'') - c\left(\frac{x''}{a_h}\right) + E[v_i(\mu(x''), \mu(x^{-i}), a_h)] - \varepsilon_2 \\ > F_1(x) - c\left(\frac{x}{a_h}\right) + E[v_i(\mu(x), \mu(x^{-i}), a_h)] \\ \Rightarrow F_1(x'') - c\left(\frac{x''}{a_h}\right) - \varepsilon_2 > F_1(x) - c\left(\frac{x}{a_h}\right) \end{aligned}$$

Then, by continuity, $\exists \delta_2 > 0$ such that $\forall x''$, $|x'' - x| > \delta_2$. Then $[x, x + \delta_2] \subset X_c$.

In either case, if $x \in X_c$, then there is an interval with some supremum b and infimum a such that $x \in (a, b) \subset X_c$.

If $b < x_\ell^*$, then there is an $x' \in \{(x_{h,*}, x_\ell^*) \cap X_1^\ell \cap X_1^h\}$ where $|x' - b| < \delta$ for all $\delta > 0$. Therefore there is an $x' \in \{(x_{h,*}, x_\ell^*) \cap X_1^\ell \cap X_1^h\}$ such that $F(x') - F(b) < c(b/a_h) - c(\frac{b+a}{2a_h})$. Note that this implies that $F_1(x') - F_1(\frac{b+a}{2}) < c(x'/a_h) - c(\frac{b+a}{2a_h})$ and $F_1(x') - F_1(\frac{b+a}{2}) < c(x') - c(\frac{b+a}{2})$.

If $\mu((b+a)/2) < \mu(x')$ then

$$\begin{aligned} E\left[\pi^i\left(\frac{b+a}{2}\right)|a_h\right] &= F_1\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2a_h}\right) + E\left[v_i\left(\mu\left(\frac{b+a}{2}\right), \mu(x^{-i}), a_h\right)\right] \\ &> F_1(x') - c\left(\frac{x'}{a_h}\right) + E[v_i(\mu(x'), \mu(x^{-i}), a_h)] = E[\pi^i(x')|a_h]. \end{aligned}$$

If $\mu((b+a)/2) \geq \mu(x')$ then

$$\begin{aligned} E\left[\pi^i\left(\frac{b+a}{2}\right)|a_\ell\right] &= F_1\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2}\right) + E\left[v_i\left(\mu\left(\frac{b+a}{2}\right), \mu(x^{-i}), a_\ell\right)\right] \\ &> F_1(x') - c(x') + E[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] = E[\pi^i(x')|a_\ell]. \end{aligned}$$

In either case, this contradicts $x' \in \{X_1^\ell \cap X_1^h\}$.

If $b = x_\ell^*$, then there is an $x' \in X_1^h$, such that $|x' - b| < \delta$, and $\mu(x') = 1$. This implies that $F_1(x') - F_1(\frac{b+a}{2}) < c(x'/a_h) - c(\frac{b+a}{2a_h})$, and

$$\begin{aligned} E\left[\pi^i\left(\frac{b+a}{2}\right)|a_h\right] &= F_1\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2a_h}\right) + E\left[v_i\left(\mu\left(\frac{b+a}{2}\right), \mu(x^{-i}), a_h\right)\right] \\ &> F_1(x') - c\left(\frac{x'}{a_h}\right) + E[v_i(\mu(x'), \mu(x^{-i}), a_h)] = E[\pi^i(x')|a_h]. \end{aligned}$$

This contradicts $x' \in X_1^h$. Therefore X_c must be empty and for all $x \in (x_{h,*}, x_\ell^*)$, we must have $x \in \{BR(a_\ell) \cap BR(a_h)\}$.

□

Corollary 4.6 *The belief function and the distribution functions of output are continuous in output on $(0, x_h^*)$. Additionally, the belief function is given by $\mu(x) = 0$ for all $x \in [0, x_{h,*}]$, $\mu(x) = 1$ for all $x \in [x_\ell^*, x_h^*]$ and is weakly increasing on $(x_{h,*}, x_\ell^*)$.*

Proof. By definition, distribution functions are right continuous. Lemma 4.3 shows that there no output is played with positive probability by either low or high ability players. This implies that the right limit of the distribution function is equal to the left limit at every point. Therefore H_1 and L_1 are continuous and $F_1 = \frac{1}{2}L_1 + \frac{1}{2}H_1$ is also continuous.

To show that $\mu(x)$ is continuous on $(0, x_h^*)$, note that $E[\pi^i(x)|a_\ell]$ is constant for all $x \in BR(a_\ell)$ and $E[\pi^i(x)|a_h]$ is constant for all $x \in BR(a_h)$. Since both $F_1(x)$ and $c(x)$ are continuous on $(0, \infty)$ and $E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] = c(x, 1) - F_1(x) + k_\ell$ on $[0, x_\ell^*]$ for some constant k_ℓ, then $E[v_i(\mu(x), \mu(x^{-i}), a_\ell)]$ must be continuous on this interval. Also, $E[v_i(\mu(x), \mu(x^{-i}), a_h)] = c(\frac{x}{a_h}) - F_1(x) + k_h$ on $[x_{h,*}, x_h^*]$ for some constant k_h, then $E[v_i(\mu(x), \mu(x^{-i}), a_h)]$ is continuous on this interval. Since $E[v_i(\mu(x), \mu(x^{-i}), a_h)]$ is strictly decreasing in $\mu(x)$, and $E[v_i(\mu(x), \mu(x^{-i}), a_\ell)]$ is strictly increasing in $\mu(x)$, then $\mu(x)$ must also be continuous on $BR(a_\ell) \cup BR(a_h) = [0, x_h^*]$.

Using the above, we now show that the set $[0, x_h^*] \setminus X_1$ has no interior, i.e. there can be no interval $[a, b] \subset [0, x_h^*]$ where for all $x \in [a, b]$, $x \notin X_1$. This implies that X_1 is dense in $[0, x_h^*]$.

If we let $[\tilde{a}, \tilde{b}] \subset [0, x_h^*] \setminus X_1$ be given, then define a and b to be the infimum and supremum respectively of the interval in $[0, x_h^*] \setminus X_1$ which contains $[\tilde{a}, \tilde{b}]$. Neither $x_{h,*}$ nor x_ℓ^* can be contained in the interval as they are the limit point of a subset of X_1. Then the interval $[a, b]$ must be contained within either $[0, x_{h,*}]$, $[x_{h,*}, x_\ell^*]$, or $[x_\ell^*, x_h^*]$.

1. If $[a, b] \subset [0, x_{h,*}]$, then for all $x \in [a, b]$, $f_1(x) = 0$ which implies that $F_1(x) = F_1(a)$ and $x \in BR(a_\ell)$. Therefore,

$$E[v_i(\mu(b), \mu(x^{-i}), a_\ell)] - c(b) = E[v_i(\mu(a), \mu(x^{-i}), a_\ell)] - c(a),$$

and $\mu(b) > \mu(a)$. Then for all $\delta > 0$, there is an $x \in X_1$ such that $|x - b| < \delta$. If $x \in X_1^\ell$, we must have $\mu(x) = 0$ or $x \in X_1^h$. Since $\mu(x)$ is continuous, then

$\mu(x) \neq 0$, so $x \in X_1^h$. Also if $x \in X_1^\ell$, then $x \in X_1^h$. Either way, for all $\delta > 0$, there must be an $x \in X_1^h$ for which $|x - b| < \delta$. If $x \in X_1^h \setminus X_1^\ell$, then $\mu(x)$=1, and $E[\pi^i(\frac{a+b}{2})|a_h] > E[\pi^i(x)|a_h]$, a contradiction. If $x \in X_1^h \cap X_1^\ell$ then either $E[\pi^i(\frac{a+b}{2})|a_\ell] > E[\pi^i(a)|a_\ell]$ or $E[\pi^i(\frac{a+b}{2})|a_h] > E[\pi^i(x)|a_h]$, again a contradiction. Therefore $[a, b] \not\subset [0, x_{h,*}]$.

2. If $[a, b] \subset [x_{h,*}, x_\ell^*]$, then for all $x \in [a, b]$, $x \in \{BR(a_\ell) \cap BR(a_h)\}$ which implies

$$E[v_i(\mu(b), \mu(x^{-i}), a_\ell)] - c(b) = E[v_i(\mu(a), \mu(x^{-i}), a_\ell)] - c(a),$$
$$E[v_i(\mu(b), \mu(x^{-i}), a_h)] - c(b/a_h) = E[v_i(\mu(a), \mu(x^{-i}), a_h)] - c(a/a_h)$$

This gives

$$E[v_i(\mu(b), \mu(x^{-i}), a_\ell)] - E[v_i(\mu(a), \mu(x^{-i}), a - \ell)] = c(b) - c(a) > 0,$$
$$E[v_i(\mu(b), \mu(x^{-i}), a_h)] - E[v_i(\mu(a), \mu(x^{-i}), a_h)] = c(b/a_h) - c(a/a_h) > 0.$$

However, these inequalities cannot hold at the same time, so $[a, b] \not\subset [x_{h,*}, x_\ell^*]$.

3. If $[a, b] \subset [x_\ell^*, x_h^*]$, then for all $x \in [a, b]$, $x \in BR(a_h)$ and therefore,

$$E[v_i(\mu(b), \mu(x^{-i}), a_h)] - c(b/a_h) = E[v_i(\mu(a), \mu(x^{-i}), a_h)] - c(a/a_h),$$

and $\mu(b) < \mu(a) \leq 1$. Then for all $\delta > 0$, there is an $x \in X_1^h$ such that $|x - b| < \delta$ and $\mu(x) = 1$. However, this contradicts the continuity of $\mu(x)$. Therefore $[a, b] \not\subset [x_\ell^*, x_h^*]$.

Therefore the interior of $[0, x_h^*] \setminus X_1$ is empty, and X_1 is dense on $[0, x_h^*]$.

Since X_1 is dense on $[0, x_h^*]$ we can now show that $\mu(x) = 0$ for any $x \in [0, x_h^*)$. If $\mu(x) = \varepsilon > 0$, then by the continuity of $\mu(x)$, $\exists \delta > 0$ where $\forall x', |x' - x| < \delta$, $\mu(x) > \varepsilon/2$. However for all $\delta > 0$ there is an $x' \in X_1^\ell \setminus X_1^h$ for which $\mu(x') = 0$, a contradiction. Therefore $\mu(x) = 0$ for all $x \in [0, x_h^*)$. Note that $\mu(x_h^*) = 0$ which follows from a similar argument of continuity form the left. Additionally, $\mu(x) = 1$ for all $x \in [x_\ell^*, x_h^*]$.

Lastly we show that $\mu(x)$ is weakly increasing on $[x_{h,*}, x_\ell^*]$. Let $x, y \in [x_{h,*}, x_\ell^*]$ be such that, $\mu(x) > \mu(y)$ and $x < y$. Then there is an x' and y' arbitrarily close to x and y respectively, where $x', y' \in X_1$ and therefore $\mu(x') \leq \mu(y')$. This is not consistent with $\mu(\cdot)$ being continuous, a contradiction. □

Theorem 4.3

Proof. The lemmas above show that there are three distinct intervals in each equilibrium. We will show that the endpoints of these intervals and the distribution functions on the intervals are completely determined by the first order conditions of players.

The three intervals we investigate are partitioned by the best response sets of the high and low ability players. The first is the set of outputs where only low ability players are optimizing: $[0, x_{h,*}) = \{BR(a_\ell) \setminus BR(a_h)\}$. Next is the set of outputs where both high and low ability players are optimizing $[x_{h,*}, x_\ell^*] = \{BR(a_\ell) \cap BR(a_h)\}$. Lastly is the set of outputs where only high ability players are optimizing: $(x_\ell^*, x_h^*] =$

$\{BR(a_h) \setminus BR(a_\ell)\}$. For each output where $x \in BR(a_\ell)$, the low ability player's first order condition must hold and likewise for each $x \in BR(a_h)$ the high ability player's first order condition must hold.

Conditions for x being in $BR(a_h)$ and $BR(a_\ell)$ are

$$BR(a_h):\ F_1^*(x) + E[v_i(\mu(x), \mu(x^{-i}), a_h)] - c\left(\frac{x}{a_h}\right) = k_h$$

$$BR(x_\ell):\ F_1^*(x) + E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] - c(x) = k_\ell = 0$$

For the range of $0 \leq x < x_{h,*}$ we have that $E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] = 0$ as $\mu(x) = 0$. Therefore we have that $F_1^*(x) = c(x)$ for all $x \in [0, x_{h,*}]$.

For the range $x_\ell^* < x \leq x_h^*$, $E[v_i(\mu(x), \mu(x^{-i}), a_h)] = E_{x_j}[v_i(1, \mu(x_j), a_h)] \equiv v_h$. Then we have $F_1^*(x) + v_h = c(x/a_h) + k_h$, for all $x \in [x_\ell^*, x_h^*]$.

For the range $x_{h,*} \leq x \leq x_\ell^*$, for all $x \in \{X_1^\ell \cup X_1^h\}$ we know $x \in \{X_1^\ell \cap X_1^h\}$. Therefore, both low and high ability players are indifferent between all outputs in this range. Because the marginal cost of the low ability player is always more than the marginal cost of the high ability player, this can only be true if increasing output benefits the low ability player more than the high ability player. Since the beliefs players have in the second period completely determine their expected payoffs, this indifference condition determines the belief function over this interval. The difference in marginal benefits of increasing output for the high ability and low ability players must equal the difference in marginal costs that they face today. To derive this, we subtract the condition for X_1^ℓ from the condition for X_1^h.

$$E[v_i(\mu(x), \mu(x^{-i}), a_h)] - E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] = c\left(\frac{x}{a_h}\right) + k_h - c(x)$$

Taking the derivative of each side with respect to output,

$$\begin{aligned}
\frac{\partial}{\partial x}(E[v_i(\mu(x), \mu(x^{-i}), a_h)] - E[v_i(\mu(x), \mu(x^{-i}), a_\ell)]) &= \frac{\partial}{\partial x}(c\left(\frac{x}{a_h}\right) - c(x)) \\
\mu'(x)\left[d(\mu(x))(F_\mu(\mu(x)) - 1) - d(\mu(x))F_\mu(\mu(x))\right] &= \frac{1}{a_h}c'\left(\frac{x}{a_h}\right) - c'(x) \\
\mu'(x)d(\mu(x)) &= c'(x) - \frac{1}{a_h}c'(x/a_h)
\end{aligned}$$

Note that on this interval, $\mu'(x) > 0$ and therefore, $F_\mu(\mu(x)) = F_1^*(x)$ for all $x \in (x_{h,*}, x_\ell^*)$.

We now take the derivative of the condition for X_1^ℓ and combine with the previous equality:

$$\begin{aligned}
f_1^*(x) + \mu'(x)d(\mu(x))F_1^*(x) &= c'(x) \\
f_1^*(x) + \left(c'(x) - \frac{1}{a_h}c'\left(\frac{x}{a_h}\right)\right)F_1^*(x) &= c'(x) \\
f_1^*(x) &= \frac{\partial}{\partial x}c(x)(1 - F_1^*(x)) + \frac{\partial}{\partial x}c\left(\frac{x}{a_h}\right)F_1^*(x) \quad (\dagger)
\end{aligned}$$

From continuity of $F_1^*(x)$, we also have that $F_1^*(x_{h,*}) = c(x_{h,*})$. For a given $x_{h,*}$, using the Picard - Lindelöf Theorem[15], we know that there is a unique solution for $f_1^*(x)$ on $[x_{h,*}, x_\ell^*]$, and therefore $F_1^*(x)$ is determined on this interval. Additionally, given $f_1(x)$ on $[0, x^*]$, the endpoints $x_{h,*}$, x_ℓ^*, and x_h^* can be solved for. With $\mu(x)$ characterized, and the equilibrium strategies of high ability and low ability players can be calculated.

To see why only one such $x_{h,*}$ can lead to an equilibrium, consider a different initial condition, $F^*(\tilde{x}_{h,*}) = c(\tilde{x}_{h,*})$ where $\tilde{x}_{h,*} > x_{h,*}$ and the associated $\tilde{f}_1(x)$ on $[\tilde{x}_{h,*}, \tilde{x}_\ell^*]$. First note that $L(\tilde{x}_{h,*}) = c(\tilde{x}_{h,*}) > c(x_{h,*}) = L(x_{h,*})$. Also, from (†), for each $x \in [\tilde{x}_{h,*}, \tilde{x}_\ell^*]$, $\tilde{f}_1(x) > f_1(x)$. Lastly, $\tilde{\mu}(\tilde{x}_{h,*}) < \mu(\tilde{x}_{h,*})$. Since $\mu(x) = 1 - \frac{\ell(x)}{2f(x)}$, then for all x where $\tilde{\mu}(x) < \mu(x)$, we must have $\tilde{\ell}(x) > \ell(x)$. Then we have that $\tilde{\ell}(x) > \ell(x)$ and $\tilde{\mu}(x) < \mu(x)$ for every $x \in [\tilde{x}_{h,*}, \tilde{x}_{x,*} + \varepsilon]$. In order to get $\tilde{L}(\tilde{x}_\ell^*) = \tilde{\mu}(\tilde{x}_\ell^*) = 1$, there must be an x such that $\tilde{f}(x) = f(x)$ in $[\tilde{x}_{h,*}, \tilde{x}_\ell^*]$, but this cant be true because $\tilde{f}(x)$ and $f(x)$ are different members of the same family of solutions, and cannot cross. Similarly, there cannot be an equilibrium where $\tilde{x}_{h,*} < x_{h,*}$.

Therefore $F_1^*(x)$ is uniquely characterized on X_1 where $\overline{X}_1 = [0, x_h^*]$. Then $L_1^*(x)$ and $H_1^*(x)$ are uniquely determined on this set. These distributions along with the second period output distributions $L_2^*(x|\mu_i, \mu_{-i})$ and $H_2^*(x|\mu_i, \mu_{-i})$ form the unique symmetric Bayes Nash equilibrium. □

References

Aoyagi, Masaki (2010), "Information feedback in a dynamic tournament." *Games and Economic Behavior*, 70, 242–260.

Avery, Christopher (1998), "Strategic jump bidding in english auctions." *The Review of Economic Studies*, 65, 185–210.

Che, Yeon-Koo and Ian Gale (2003), "Optimal design of research contests." *The American Economic Review*, 93, 646–671.

Ederer, Florian (2010), "Feedback and motivation in dynamic tournaments." *Journal of Economics & Management Strategy*, 19, 733–769.

Fullerton, Richard L and Randolph Preston McAfee (1999), "Auctioning entry into tournaments." *Journal of Political Economy*, 107, 573–605.

Gallini, Nancy and Suzanne Scotchmer (2002), "Intellectual property: when is it the best incentive system?" In *Innovation Policy and the Economy, Volume 2*, 51–78, MIT Press.

Harris, Christopher and John Vickers (1987), "Racing with uncertainty." *The Review of Economic Studies*, 54, 1–21.

[15] The right hand side of (†) is continuous in x and uniformly Lipshitz continuous in $F_1^*(x)$ on the interval of $[x_{h,*}, x_\ell^*]$, see Lindelöf (1894)

Hörner, Johannes and Nicolas Sahuguet (2007), "Costly signalling in auctions." *The Review of Economic Studies*, 74, 173–206.

Konrad, Kai A (2012), "Dynamic contests and the discouragement effect." *Revue d'économie politique*, 122, 233–256.

Kräkel, Matthias (2014), "Sandbagging." *Journal of Sports Economics*, 15, 263–284.

Lindelöf, Ernest (1894), "Sur l'application des méthodes d'approximations successives à l'étude des intégrales réelles des équations différentielles ordinaires." *Journal de Mathématiques Pures et Appliquées*, 117–128.

Mirman, Leonard J, Larry Samuelson, and Amparo Urbano (1993), "Duopoly signal jamming." *Economic Theory*, 3, 129–149.

Moldovanu, Benny and Aner Sela (2006), "Contest architecture." *Journal of Economic Theory*, 126, 70–96.

Münster, Johannes (2009), "Repeated contests with asymmetric information." *Journal of Public Economic Theory*, 11, 89–118.

Ortega Reichert, Armando (2000), "A sequential game with information flow." In *The Economic Theory of Auctions I*, 232–54, Edward Elgar Publishing Limited.

Ridlon, Robert and Jiwoong Shin (2013), "Favoring the winner or loser in repeated contests." *Marketing Science*, 32, 768–785.

Rosen, Sherwin (1986), "Prizes and incentives in elimination tournaments." *The American Economic Review*, 701–715.

Siegel, Ron (2009), "All-pay contests." *Econometrica*, 77, 71–92.

Siegel, Ron (2014), "Asymmetric all-pay auctions with interdependent valuations." *Journal of Economic Theory*, 153, 684–702.

Ye, Lixin (2007), "Indicative bidding and a theory of two-stage auctions." *Games and Economic Behavior*, 58, 181–207.